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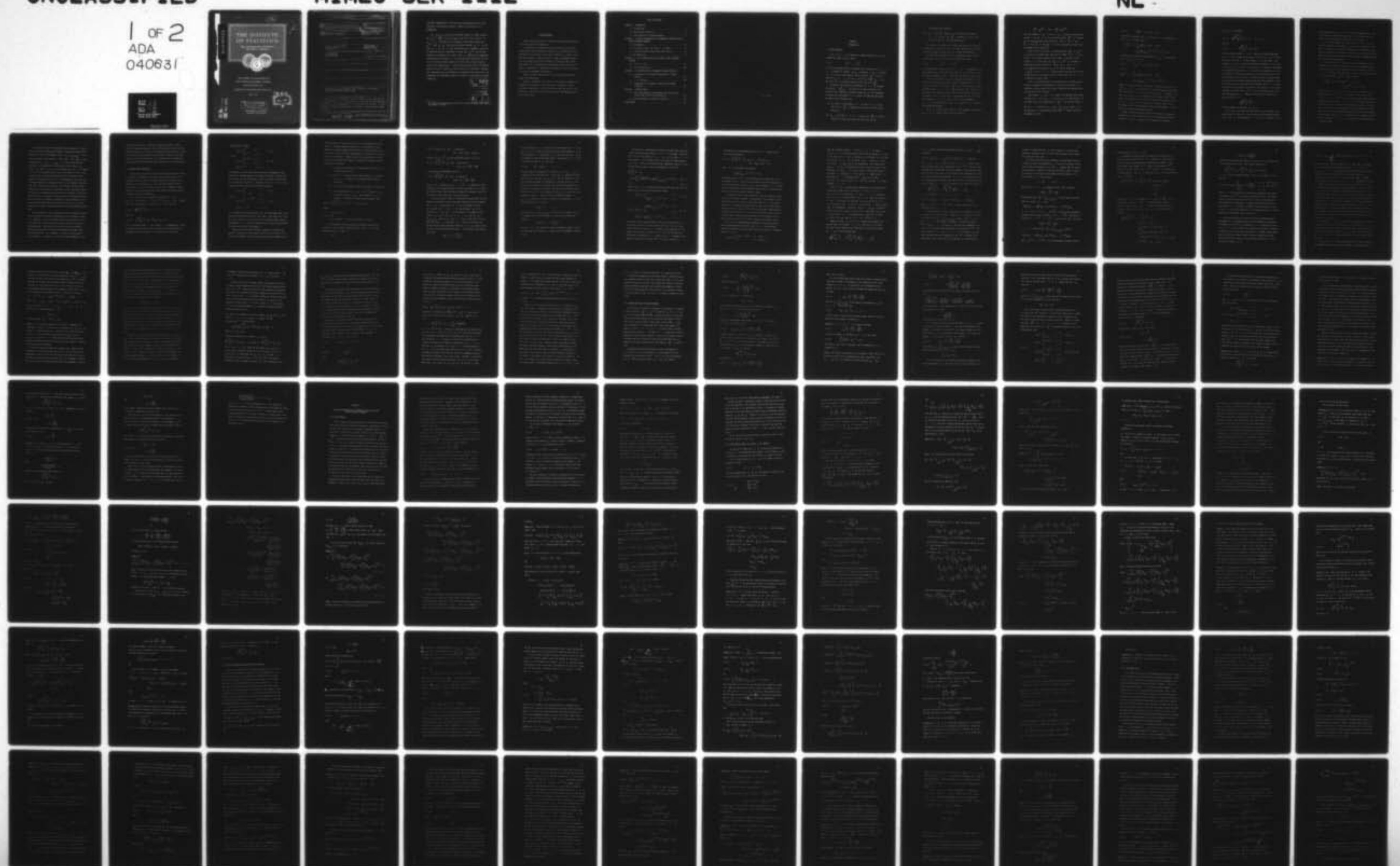

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THE INSTITUTE OF STATISTICS

THE CONSOLIDATED UNIVERSITY
OF NORTH CAROLINA

LIMIT THEOREMS AND REPRESENTATIONS FOR
ORDER STATISTICS FROM DEPENDENT SEQUENCES

John Henry Vernon Watts

INSTITUTE OF STATISTICS MIMEO SERIES #1112

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Let $\{\xi_n\}_{n\geq 1}$ be a stationary sequence of random variables and $\xi_{k_n}^{(n)}$ the k_n-th smallest of $\xi_1,\ldots,\xi_n$. Considered is the asymptotic distribution problem for $\xi_{k_n}^{(n)}$ when k_n is fixed and when $k_n\to\infty$. For the case $k_n\to\infty$ but $kn/n \to 0$, representations for $\xi_{k_n}^{(n)}$ are obtained which are analogues of those known for sample λ-quantiles.

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JOHN HENRY VERNON WATTS.* Limit Theorems And Representations For Order Statistics From Dependent Sequences. (Under the direction of M. R. LEADBETTER.)

Let $\{\xi_n\}_{n\geq 1}$ be a strictly stationary sequence of random variables and ${}_{k_n}\xi^{(n)}$ $(\xi^{(n)}_{k_n})$ the k_n-th largest (smallest) order statistic for $\xi_1, \ldots, \xi_n$. Considered is the asymptotic distribution problem for ${}_{k_n}\xi^{(n)}$ when k_n is a fixed positive integer and when $k_n \to \infty$. For the case $k_n \to \infty$ but $\frac{k_n}{n} \to 0$ (the intermediate case), dependence conditions are obtained which are sufficient to insure that ${}_{k_n}\xi^{(n)}$ has the same asymptotic distribution as it would if the ξ_n sequence were independent and identically distributed. It is shown that these conditions are satisfied by stationary normal sequences for which the covariance function tends to zero exponentially. A multivariate extension of the distribution problem to several sequences of order statistics is considered. Additionally, representations are developed for intermediate order statistics from independent and m-dependent sequences, analogous to those known for sample λ-quantiles.

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to my parents

CHAPTER I

INTRODUCTION

1.1 Preliminaries

Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables (r.v.'s) on a probability space (Ω, F, P), and let

$$\xi_1^{(n)} \le \ldots \le \xi_n^{(n)} \tag{1.1.1}$$

be the corresponding series of ascending order statistics for $\xi_1, \ldots, \xi_n$. A sequence of integers $\{k_n\}_{n\ge 1}$ satisfying $1 \le k_n \le n$ for each $n \ge 1$ is called a rank sequence, and the variable $\xi_{k_n}^{(n)}$, the k_n-th smallest of $\xi_1, \ldots, \xi_n$, we say is of rank k_n among $\xi_1, \ldots, \xi_n$. The ratio $\frac{k_n}{n}$ is called the relative rank of the term $\xi_{k_n}^{(n)}$.

We consider the asymptotic distributions as $n \to \infty$ of sequences of variables $\{\xi_{k_n}^{(n)}\}_{n\ge 1}$. The basis of our investigation is the theory of such asymptotic distributions when the ξ_n are independent and identically distributed (i.i.d.), which is extensively developed for each of the following cases:

(i) k_n equals a fixed integer $k \ge 1$ for each $n \ge k$. In this case we call $\xi_{k_n}^{(n)}$ a sequence of *fixed rank* or *extreme* order statistics.

(ii) $k_n \to \infty$ but $\frac{k_n}{n} \to 0$ as $n \to \infty$. In this case $\xi_{k_n}^{(n)}$ is called a sequence of *intermediate* order statistics and k_n an

intermediate rank sequence.

(iii) $k_n \to \infty$ and $\frac{k_n}{n}$ converges to a limiting relative rank λ, $0 < \lambda < 1$. Here we call $\xi_{k_n}^{(n)}$ a sequence of *central* order statistics and k_n a central rank sequence.

We refer to (i) as the case of fixed ranks and to (ii) and (iii) collectively as the case of *increasing* ranks, although for the latter we do not in general require that k_n increase monotonically.

Of course the arrangement (1.1.1) in ascending order is merely conventional, and we may just as well consider the series

$$ {}_1\xi^{(n)} \geq \ldots \geq {}_n\xi^{(n)} \tag{1.1.2} $$

of descending order statistics for $\xi_1, \ldots, \xi_n$. Correspondingly, for rank sequences k_n we have the same cases (i), (ii), and (iii), with the same terminology extreme, intermediate, and central, respectively.

Given a rank sequence k_n, to distinguish the sequence of order statistics resulting from the two possible arrangements (1.1.1) and (1.1.2), we may say the variables $\xi_{k_n}^{(n)}$ are of left or smallest rank, and the variables ${}_{k_n}\xi^{(n)}$ are of right or largest rank. Moreover, we may identify the arrangement with which we are dealing by calling k_n a sequence of left or right ranks. In particular we call $\xi_k^{(n)}$ the k-th minimum of $\xi_1, \ldots, \xi_n$, and ${}_k\xi^{(n)}$ the k-th maximum of $\xi_1, \ldots, \xi_n$, respectively, for each $n \geq k$, and for $k = 1$ we merely say minimum and maximum.

If k_n is a rank sequence, then so is the sequence k_n' given by $k_n' = n - k_n + 1$, and we clearly have the relations

$$\xi_{k_n}^{(n)} = {}_{k'_n}\xi^{(n)} \qquad \text{and} \qquad \xi_{k'_n}^{(n)} = {}_{k_n}\xi^{(n)} .$$

Thus, for example, if $k_n \to \infty$ but $n - k_n + 1$ equals a fixed positive integer k for all $n \geq k$, we may actually regard $\xi_{k_n}^{(n)}$ as a sequence of k-th largest extreme order statistics. Also, if $k_n \to \infty$ and $\frac{k_n}{n} \to 1$, then $\frac{k'_n}{n} \to 0$, and $\xi_{k_n}^{(n)}$ is in fact a sequence of largest rank intermediate order statistics.

Hence we are led to state as a definition that $\xi_{k_n}^{(n)}$ or ${}_{k_n}\xi^{(n)}$ is a sequence of extreme order statistics if either k_n or $n - k_n + 1$ equals a fixed positive integer for all large n, and that a rank sequence k_n is *increasing*, and $\xi_{k_n}^{(n)}$ or ${}_{k_n}\xi^{(n)}$ is a sequence of increasing rank order statistics, if both $k_n \to \infty$ and $n - k_n \to \infty$. For increasing rank sequences k_n we will always assume that $\lambda = \lim_n \frac{k_n}{n}$ exists, $0 \leq \lambda \leq 1$. Other possible cases which we will not consider include, for example, those for which either k_n or $n - k_n$ is bounded but without limit.

Since, as previously noted, our investigation is founded upon the problem of asymptotic distributions of order statistics from i.i.d. sequences, we now introduce the further terminology and concepts which are relevant to that situation.

Thus suppose ξ_n is an i.i.d. sequence with marginal distribution function (d.f.) $F(x) = P(\xi_1 \leq x)$, and let k_n be a rank sequence. We can easily obtain the distribution of $\xi_{k_n}^{(n)}$: For each real number x and for $n \geq 1$ define the r.v. $T_n(x) = \sum_{i=1}^{n} I_{[\xi_i \leq x]}$, where I_E is the indicator r.v. corresponding to the event E. Then we have the fundamental relation

$$P\left(\xi_{k_n}^{(n)} \le x\right) = P\left(T_n(x) \ge k_n\right), \tag{1.1.3}$$

from which, regarding $T_n(x)$ as a binomial r.v. with parameters n and $F(x)$, we readily obtain

$$\begin{aligned} P\left(\xi_{k_n}^{(n)} \le x\right) &= \sum_{r=k_n}^{n} \frac{n!}{r!(n-r)!} F^r(x)\left(1 - F(x)\right)^r \\ &= \frac{n!}{(k_n-1)!(n-k_n)!} \int_0^{F(x)} y^{k_n-1} (1-y)^{n-k_n}\, dy. \end{aligned}$$

Similarly we can obtain the distribution of ${}_{k_n}\xi^{(n)}$: Now writing $W_n(x) = \sum_{i=1}^{n} I_{[\xi_i > x]}$ we have

$$P\left({}_{k_n}\xi^{(n)} \le x\right) = P\left(W_n(x) < k_n\right), \tag{1.1.4}$$

so that

$$\begin{aligned} P\left({}_{k_n}\xi^{(n)} \le x\right) &= \sum_{r=0}^{k_n-1} \frac{n!}{r!(n-r)!} F^{n-r}(x)\left(1 - F(x)\right)^r \\ &= \frac{n!}{(k_n-1)!(n-k_n)!} \int_0^{F(x)} y^{n-k_n}(1-y)^{k_n-1}\, dy. \end{aligned}$$

However, it turns out that in situations of interest it is not the r.v. $\xi_{k_n}^{(n)}$ (or ${}_{k_n}\xi^{(n)}$) itself which has a limit distribution, but rather some normalization of it, or more precisely, the r.v. $a_n^{-1}\left(\xi_{k_n}^{(n)} - b_n\right)$, where $\{a_n > 0,\ b_n;\ n \ge 1\}$ are sequences of real numbers. This corresponds in an analagous way to the usual central limit theory for sums $S_n = \xi_1 + \ldots + \xi_n$ of r.v.'s, where S_n in general has no limit distribution itself, but may under suitable standardization. In what follows we consider this notion formally.

Perhaps the most important questions in the study of asymptotic distributions of order statistics from i.i.d. sequences involve the concept of *attraction*. Suppose for some sequences of constants

$a_n > 0$, b_n we have that

$$(1.1.5) \qquad \mathcal{L}\left(\frac{\xi^{(n)}_{k_n} - b_n}{a_n}\right) \to \text{d.f. } G \quad \text{as} \quad n \to \infty,$$

that is,

$$(1.1.6) \qquad P\left(\frac{\xi^{(n)}_{k_n} - b_n}{a_n} \le x\right) \to G(x)$$

for all continuity points x of G. Then we say that the marginal d.f. F of ξ_1 belongs to the *domain of attraction* of, or is attracted to, the limit law G for the sequence k_n of left ranks. The sequences $a_n > 0$, b_n are called normalizing constants. As a particular case, if $k_n = 1$ for all n, we say that F belongs to the domain of attraction for minima of G. For a given rank sequence, the classical problem of limiting distributions of order statistics consists of first finding the class of laws G which do possess a nonvoid domain of attraction, and secondly finding conditions which completely characterize the domain of attraction of each possible limit law. Of course the concept of attraction also applies to the problem of limit laws for sequences of largest terms ${}_{k_n}\xi^{(n)}$, and as a particular case, we say that the marginal d.f. F belongs to the domain of attraction for maxima of the law G if

$$\mathcal{L}\left(\frac{{}_1\xi^{(n)} - b_n}{a_n}\right) \to G$$

for some sequence of constants $a_n > 0$, b_n.

It is easy to see that each limit law is in fact a member of an entire "type" of limit laws. By considering the transformed sequence $\xi'_n = a^{-1}(\xi_n - b)$ for real numbers $a > 0$, b, it is clear that if for a

given rank sequence k_n, $G(x)$ is a limit law having the d.f. $F(x)$ in its domain of attraction, then $G(ax + b)$ is also a limit law, whose domain of attraction includes the d.f. $F(ax + b)$.

For each rank sequence k_n and for any d.f. $F(x)$ it can be shown, by choosing the constants a_n to increase sufficiently fast, that F belongs to the domain of attraction of each degenerate law, so that the degenerate laws contain the entirety of distributions in their domains of attraction. Thus we exclude convergence to degenerate limit laws from further consideration.

At least for the i.i.d. case, it may be seen that all results obtained for k_n-th smallest terms can easily be transformed into corresponding results for k_n-th largest terms, and conversely. Letting $\eta_i = -\xi_i$ we trivially have that the η_n are i.i.d. if and only if the ξ_n are, and moreover, ${}_{k_n}\eta^{(n)} = -\xi_{k_n}^{(n)}$, so that

$$P\left(\xi_{k_n}^{(n)} \le a_n x + b_n\right) = P\left({}_{k_n}\eta^{(n)} \ge -a_n x - b_n\right)$$

$$= 1 - P\left({}_{k_n}\eta^{(n)} < a_n(-x) - b_n\right).$$

Thus for example, if $G(x)$ is a limit law for k_n-th smallest terms, which contains the d.f. $F(x)$ in its domain of attraction, then $1 - G(-x - 0)$ is a limit law for k_n-th largest terms, whose domain of attraction contains the d.f. $1 - F(-x - 0)$. This simple relationship permits us to freely transfer our attention from smallest to largest terms, and vice versa. Additionally, it turns out that all possible limit laws in the extreme and intermediate ranks cases are continuous, so that in those cases $G(x)$ is a limit law for a sequence of largest ranks if and only if $1 - G(-x)$ is a limit law for the same sequence of smallest ranks.

We mention here that the framework we have described is a special case of a general "ranking limit problem" formulated by Loève (1956). Considered in this are triangular arrays $\{\xi_{n,j}\}_{n\geq 1, 1\leq j\leq m_n \to\infty}$ of r.v.'s and the asymptotic distributions of terms $\xi_{k_n}^{(n)}$, where $\xi_{k_n}^{(n)}$ is the k_n-th smallest of $\xi_{n,1}, \ldots, \xi_{n,m_n}$. It is easily seen that the situation we are dealing with is the special case in which $\xi_{n,j} = a_n^{-1}(\xi_j - b_n)$, where a_n, b_n are real sequences and $\{\xi_j\}$ is a sequence of r.v.'s. This generalization of course corresponds to, and in fact is motivated by, the familiar generalization of the central limit theorem for normalized sums of r.v.'s to convergence of row sums from uniformly asymptotically negligible arrays to the infinitely divisible limit laws. However we have chosen not to examine this more general setting in our investigation; nevertheless, we have found that Loève's results can be conveniently specialized to our situation of normalized order statistics of sequences, and in particular, his results dealing with the problem of finding the asymptotic joint distributions of two or more extreme order statistics, which even in the i.i.d. case has apparently not received substantial attention in other published literature.

In the remainder of this introductory chapter we present the main results dealing with limiting distributions of order statistics from i.i.d. sequences. We first consider extreme order statistics, or the fixed ranks case, in the next section, and in Section 1.3 we turn to increasing ranks. Additionally in the next section we introduce a highly developed theory of limiting distributions of extreme order statistics when the i.i.d. assumption is relaxed, and more specifically, when the ξ_n sequence is allowed to exhibit dependence of the

type involving "mixing." From this we might expect that a similar theory for dependent sequences could be developed for increasing ranks, and in particular, for the case of intermediate order statistics, which in some sense can be regarded as being "close" to that of fixed ranks, among the increasing ranks cases. The consideration of intermediate order statistics will thus become the focal point of our investigation.

1.2 Extreme Order Statistics.

In this section we sketch the main results dealing with the theory of asymptotic distributions of extreme order statistics from i.i.d. sequences and from a class of dependent sequences to which the techniques and results from the i.i.d. case have been found to conveniently extend. In discussing these results we will primarily consider maxima or largest fixed ranks rather than minima, since this approach is taken in most of the literature.

Let ξ_n be an i.i.d. sequence with marginal d.f. $F(x) = P(\xi_1 \leq x)$, and for each $n \geq 1$ let ${}_1\xi^{(n)} = \max\{\xi_1, \ldots, \xi_n\}$. Suppose that there are sequences of constants $a_n > 0$, b_n such that

$$(1.2.1) \qquad \mathcal{L}\left(\frac{{}_1\xi^{(n)} - b_n}{a_n}\right) \to \text{d.f. } G \quad \text{as} \quad n \to \infty;$$

that is,

$$(1.2.2) \qquad P\left(\frac{{}_1\xi^{(n)} - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x)$$

for all continuity points x of G, where G is nondegenerate. Then by the well-known classical result of Fisher and Tippett (1928) and Gnedenko (1943), the limit law G is necessarily of one of the

following three "types":

$$
(1.2.3)\qquad
\begin{aligned}
G_1(x) &= \begin{cases} 0 & , \quad x \le 0 \\ e^{-x^{-a}} & , \quad x > 0 \end{cases} \qquad (a > 0) \\
G_2(x) &= \begin{cases} e^{-(-x)^a} & , \quad x < 0 \\ 1 & , \quad x \ge 0 \end{cases} \qquad (a > 0) \\
G_3(x) &= e^{-e^{-x}} \quad , \quad -\infty < x < \infty .
\end{aligned}
$$

Put another way, these three types are the only nondegenerate laws possessing a nonempty domain of attraction for maxima. We note that all of these laws are continuous, so that in the case of attraction, (1.2.2) holds for all real x. Replacing $G(x)$ by $1 - G(-x)$ we see that the possible limit laws for minima are given by

$$
(1.2.4)\qquad
\begin{aligned}
G_1(x) &= \begin{cases} 0 & , \quad x \le 0 \\ 1 - e^{-x^{a}} & , \quad x > 0 \end{cases} \qquad (a > 0) \\
G_2(x) &= \begin{cases} 1 - e^{-(-x)^a} & , \quad x \le 0 \\ 1 & , \quad x > 0 \end{cases} \qquad (a > 0) \\
G_3(x) &= 1 - e^{-e^{x}} \quad , \quad -\infty < x < \infty .
\end{aligned}
$$

It is perhaps worth mentioning that, as is the traditional usage in the study of extreme order statistics, the "type" of the limit laws $G_1(x)$ and $G_2(x)$ given in (1.2.3) and (1.2.4) is preserved under changes in the parameter $a > 0$ as well as under the usual linear transformations (location and scale) of the argument x.

Necessary and sufficient conditions completely specifying the domains of attraction of the possible limit laws (1.2.3) and (1.2.4) are given by Gnedenko (1943), and more recently other conditions have

been obtained by de Haan (1970, 1971) (see also Balkema and de Haan (1972)). Roughly speaking, membership of a distribution in some domain of attraction depends on the behavior of the distribution near its (not necessarily finite) right endpoint, and similarly, for minima, near its left endpoint. Rather than give these conditions in detail here, we only provide some examples of attraction for maxima; similar results hold for minima.

(i) Each limit law belongs to its own domain of attraction (cf. de Haan (1970, p. 61)).

(ii) Any d.f. F having a finite right endpoint x_0 such that $F(x_0 - 0) < 1$ belongs to no (nondegenerate) domain of attraction.

(iii) The Poisson distributions belong to no domain of attraction.

(iv) The standard normal d.f. $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\frac{1}{2}y^2)dy$ is attracted to the law $G_3(x)$. Specifically, if ξ_n is a sequence of independent standard normal variables, then

$$P\left[\frac{{}_1\xi^{(n)} - b_n}{a_n} \le x\right] \to e^{-e^{-x}} \quad \text{for all real } x, \tag{1.2.5}$$

where

$$a_n = (2 \log n)^{-\frac{1}{2}}$$

(1.2.6) and

$$b_n = (2 \log n)^{\frac{1}{2}} - \tfrac{1}{2}(2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi).$$

For convergence of maxima we have the following easily proven relation (see, for example, Gnedenko (1943)), where x is a real number such that $0 < G(x) \le 1$:

$$(1.2.7)\quad F^n(a_nx + b_n) \to G(x) \quad \text{if and only if} \quad n[1 - F(a_nx + b_n)] \to -\log G(x).$$

Writing $G(x) = e^{-\tau(x)}$ we may equivalently express (1.2.7) as

$$(1.2.8)\quad P\left[\frac{{}_1\xi^{(n)} - b_n}{a_n} \le x\right] \to G(x) \quad \text{if and only if} \quad 1 - F(a_nx + b_n) = \frac{\tau(x)}{n} + o\left(\frac{1}{n}\right).$$

For minima the corresponding relation is

$$(1.2.9)\quad P\left[\frac{\xi_1^{(n)} - b_n}{a_n} \le x\right] \to G(x) \quad \text{if and only if} \quad F(a_nx + b_n) = \frac{\tau(x)}{n} + o\left(\frac{1}{n}\right),$$

where $\tau(x)$ is defined by $G(x) = 1 - e^{-\tau(x)}$. In addition to their usefulness in the generalizations to other fixed ranks $(k > 1)$, which we consider below, these relations correspond to analagous results for the cases of order statistics of increasing rank.

A problem related to that of limiting distributions deals with the number of exceedances of an increasingly "high level" by the sequence of r.v.s ξ_n. Specifically, let x be a real number for which (1.2.2) holds with $0 < G(x) \le 1$ for some sequences $a_n > 0$, b_n. Define the r.v. $W_n = W_n(x)$ to be the number of exceedances of the "level" $u_n = u_n(x) = a_nx + b_n$ by $\xi_1, \ldots, \xi_n$, that is, $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$, and let $p_n = p_n(x) = 1 - F(u_n)$. Then from (1.2.8) we have that $np_n \to \tau = \tau(x)$, where $\tau(x) = -\log G(x)$, so that by the familiar Poisson limit theorem for binomial r.v.'s, W_n asymptotically has the Poisson distribution with mean τ, a fact which we may express by writing

$$P(W_n < k) \to e^{-\tau} \sum_{r=0}^{k-1} \frac{\tau^r}{r!},$$

for each integer $k \geq 1$. Of course this result holds for all x such that $0 < G(x) \leq 1$, and in the case of attraction the sequences a_n, b_n do not depend on x. Additionally, it is clear that the level u_n may be regarded as being increasingly "high". More generally, it is easily seen to be true that if u_n satisfies

(1.2.10) $$n[1 - F(u_n)] \to \tau, \qquad 0 \leq \tau < \infty,$$

then the number of exceedances of the level u_n by $\xi_1, \ldots, \xi_n$ is asymptotically Poisson with mean τ, where the u_n need not be interpreted to be of the form $u_n = a_n x + b_n$. That is, there may be examples in which the number of exceedances of the high level u_n satisfying (1.2.10) would be of interest, but in which the marginal d.f. F belongs to no domain of attraction. We might point out however that for not all d.f.'s F can a sequence of levels u_n be so chosen (see Leadbetter (1974b)). Similarly we may consider the number of $\xi_1, \ldots, \xi_n$ which are less than or equal to an appropriate u_n level: Suppose now that u_n satisfies

$$nF(u_n) \to \tau, \qquad 0 \leq \tau < \infty.$$

Then if we let $T_n = \sum_{i=1}^{n} I_{[\xi_i \leq u_n]}$, we have in a like manner that T_n is asymptotically Poisson with mean τ, which we may conveniently express as

$$P(T_n \geq k) \to 1 - e^{-\tau} \sum_{r=0}^{k-1} \frac{\tau^r}{r!}$$

for each $k \geq 0$. This problem of course corresponds to that of limiting distributions of minima, as is clear from the fundamental relation (1.1.3).

We now turn to consideration of arbitrary extreme order statistics and see how the previous results for $k = 1$ are extended. Again let ${}_k\xi^{(n)}$ be the k-th largest of $\xi_1, \ldots, \xi_n$ $(n \ge k)$, where $k > 1$ is a fixed integer. Let us suppose that (1.2.1) holds for some sequences $a_n > 0$, b_n. From (1.2.8) along with the fundamental relation (1.1.4) and the Poisson result of the last paragraph, we find that

$$P\left\{\frac{{}_k\xi^{(n)} - b_n}{a_n} \le x\right\}$$

$$\to \begin{cases} G(x) \sum_{r=0}^{k-1} \frac{(-\log G(x))^r}{r!} = \frac{1}{(k-1)!} \int_{-\log G(x)}^{\infty} e^{-y} y^{k-1}\, dy\,, & G(x) > 0 \\ 0 & , \quad G(x) = 0, \end{cases}$$

for all real x. It is apparent that the possible limit laws for k-th largest terms include the following three types:

$$G_1^{(k)}(x) = \begin{cases} 0 & , \quad x \le 0 \\ \frac{1}{(k-1)!} \int_{x^{-a}} e^{-y} y^{k-1}\, dy & , \quad x > 0 \end{cases} \qquad (a > 0)$$

$$(1.2.11) \qquad G_2^{(k)}(x) = \begin{cases} \frac{1}{(k-1)!} \int_{(-x)^a}^{\infty} e^{-y} y^{k-1}\, dy\,, & x < 0 \\ 1 & , \quad x \ge 0 \end{cases} \qquad (a > 0)$$

$$G_3^{(k)}(x) = \frac{1}{(k-1)!} \int_{e^{-x}}^{\infty} e^{-y} y^{k-1}\, dy \qquad , \quad -\infty < x < \infty.$$

By Smirnov's (1952) generalization of the results for the case $k = 1$, these laws are in fact all the possible limit laws for k-th largest order statistics, for each $k \ge 1$. Moreover, the conditions for the domains of attraction for maxima $(k = 1)$ are also necessary and sufficient for arbitrary $k > 1$, so that if a d.f. F belongs to the domain of attraction of the law $G^{(k)}$ for some k, then it is

attracted to the corresponding law for all $k \geq 1$. These results follow from the relation

$$(1.2.12) \quad P\left\{\frac{{}_k\xi^{(n)} - b_n}{a_n} \leq x\right\} \to G(x) > 0 \quad \text{if and only if} \quad n[1 - F(a_n x + b_n)] \to v(x) ,$$

where $v(x)$ is defined by the equation

$$\frac{1}{(k-1)!}\int_{v(x)}^{\infty} e^{-y}\, y^{k-1}\, dy = G(x),$$

which generalizes (1.2.7). (For a recent proof of these facts, see Leadbetter (1977)). We note also that when we do have attraction, we may take normalizing constants a_n, b_n not depending on k. Of course, similar results hold for k-th smallest order statistics, and these may be found explicitly stated in Smirnov's paper.

Of perhaps at least as much interest as that of limiting distributions of k-th maxima or minima is the question of the asymptotic joint distributions of two or more extreme order statistics. As one might expect, it is somewhat complicated to write down such distributions in an explicit form. However, Loève (1956) has obtained a representation for these limit laws which is fairly easy to deal with, and in fact for the more general ranking problem mentioned in the previous section. Here we specialize his results to our situation (see Miller (1976)).

Let $\tau(x)$ be a nondecreasing, nonnegative, and not necessarily finite-valued function on the real line. Let μ_τ be the measure, defined on the Borel sets of the real line, whose value at a semiclosed interval $(x,y]$ is

$$\mu_\tau(x,y] = \begin{cases} \tau(y+0) - \tau(x+0) , & \text{if} \quad \tau(x+0) < \infty \\ 0 , & \text{if} \quad \tau(x+0) = \infty. \end{cases}$$

Then the stochastic process $I = \{I(x), -\infty < x < \infty\}$ is called τ-*Poisson* if I is the Poisson process with intensity measure μ_τ ; that is, for each real x, $I(x)$ is a Poisson r.v. with mean $\mu_\tau(-\infty,x]$, and for any real $x_1 \le \dots \le x_m$ for which $\tau(x_m) < \infty$, the r.v.'s $I(x_1)$, $I(x_2)-I(x_1), \dots ,I(x_m)-I(x_{m-1})$ are independent. The random vector $(I(x_1), \dots ,I(x_m))$ is called an *independent increment Poisson vector* with mean $(\mu(-\infty,x_1], \dots ,\mu(-\infty,x_m])$. Here, we define the Poisson (extended) r.v. with mean $+\infty$ to be equal to $+\infty$ almost surely. In particular, if τ is continuous and if $\lim_{x\to-\infty} \tau(x) = 0$, then $\mu_\tau(-\infty,x] = \tau(x)$ and $\mu_\tau(x,y] = \tau(y) - \tau(x)$ for all real x, y, where we define $\infty - \infty = 0$.

Now, if $\tau(x)$ is nonincreasing, nonnegative, and not necessarily finite-valued, then we call the process $J = \{J(x), -\infty < x < \infty\}$ *reverse* τ-*Poisson* if the process $I = \{I(x), -\infty < x < \infty\}$ defined by $I(x) = J(-x)$ is $\bar{\tau}$-Poisson, where $\bar{\tau}(x) = \tau(-x)$. Then, for each real x, $J(x)$ is a Poisson r.v. with mean $\mu_{\bar{\tau}}(-\infty,x]$, and for any $x_1 \le \dots \le x_m$ for which $\tau(x_1) < \infty$, the r.v.'s $J(x_1)-J(x_2), \dots ,J(x_{m-1})-J(x_m)$, $J(x_m)$ are independent. The vector $(J(x_1), \dots ,J(x_m))$ is called a *reverse independent increment Poisson vector* which mean $(\mu_{\bar{\tau}}(-\infty,x_1], \dots ,\mu_{\bar{\tau}}(-\infty,x_m])$. If τ is continuous and if $\lim_{x\to\infty} \tau(x) = 0$, then $\mu_{\bar{\tau}}(-\infty,x] = \tau(x)$ and $\mu_{\bar{\tau}}(x,y] = \tau(x) - \tau(y)$ for all x, y.

Suppose now that ξ_n is an i.i.d. sequence with marginal d.f. F and that there are constants $a_n > 0, b_n$ such that (1.2.1) holds. Then Loève (1956, Theorem 4) has shown that, for any finite number $k_1, \dots ,k_m$ of fixed largest ranks,

$$\mathcal{L}\left(\frac{{}_{k_1}\xi^{(n)} - b_n}{a_n}, \dots ,\frac{{}_{k_m}\xi^{(n)} - b_n}{a_n}\right) \to \mathcal{L}\left({}_{k_1}\xi^*, \dots ,{}_{k_m}\xi^*\right)$$

as $n \to \infty$, where the joint distribution of the r.v.'s ${}_{k_1}\xi^*, \ldots, {}_{k_m}\xi^*$ is given by

$$(1.2.13) \quad P\left({}_{k_1}\xi^* \le x_1, \ldots, {}_{k_m}\xi^* \le x_m\right) = P\left(J(x_1) < k_1, \ldots, J(x_m) < k_m\right),$$

for $-\infty < x_1 < \ldots < x_m < \infty$, where $J = \{J(x), -\infty < x < \infty\}$ is the reverse τ-Poisson process with $\tau(x)$ defined by $G(x) = e^{-\tau(x)}$. Moreover, any limiting joint distribution for a finite number of fixed largest ranks must be of this form, and as was the case for the one-dimensional limit laws, the attraction of the maximum $(k = 1)$ is also necessary for any such (nondegenerate) limiting joint law to exist. In a similar way for smallest ranks $k_1, \ldots, k_m$, the relation (1.2.9) holds if and only if

$$\mathcal{L}\left(\frac{\xi_{k_1}^{(n)} - b_n}{a_n}, \ldots, \frac{\xi_{k_m}^{(n)} - b_n}{a_n}\right) \to \mathcal{L}(\xi_{k_1}^*, \ldots, \xi_{k_m}^*)$$

as $n \to \infty$, where for $-\infty < x_1 < \ldots < x_m < \infty$

$$(1.2.14) \quad P\left(\xi_{k_m}^* \le x_1, \ldots, \xi_{k_m}^* \le x_m\right) = P\left(I(x_1) \ge k_1, \ldots, I(k_m) \ge k_m\right),$$

where $I = \{I(x), -\infty < x < \infty\}$ is the τ-Poisson process with $\tau(x)$ satisfying $G(x) = 1 - e^{-\tau(x)}$.

A result parallel to these representations is immediate, and which we state only for largest ranks. Suppose (1.2.1) holds and as defined previously, let $W_n(x) = \sum_{i=1}^{n} I_{[\xi_i > u_n(x)]}$, where $u_n(x) = a_n x + b_n$, for $-\infty < x < \infty$. This defines a sequence of stochastic processes $W^{(n)} = \{W_n(x), -\infty < x < \infty\}$, and it is clear from the fundamental relation (1.1.4) that the finite-dimensional distributions of $W^{(n)}$ converge to those of $J = \{J(x), -\infty < x < \infty\}$. It is possible that questions regarding the weak convergence of the $W^{(n)}$ processes in an appropriate space of functions on the (entire) real line would be of

interest, including when the ξ_n are no longer i.i.d. but satisfy dependence conditions which we discuss below; however we have chosen not to consider these here.

We indicate that we may also formulate a related multi-level exceedances problem, when appropriate u_n levels can be chosen, and for cases in which the marginal d.f. of the ξ_n sequence does not necessarily belong to a domain of attraction for maxima. This of course is an extension of the single level problem already discussed. For a given integer $m \geq 1$ let us choose

$$0 \leq \tau^{(m)} < \ldots < \tau^{(1)} < \infty$$

and for each $\mu = 1, \ldots, m$, suppose the level $u_n^{(\mu)}$ satisfies

$$1 - F\left(u_n^{(\mu)}\right) = \frac{\tau^{(\mu)}}{n} + o\left(\frac{1}{n}\right).$$

Then upon defining $W_n^{(\mu)} = \sum_{i=1}^{n} I_{[\xi_i > u_n^{(\mu)}]}$, we have that for any non-negative integers $k_1, \ldots, k_m$,

$$P\left(W_n^{(1)}=k_1, \ldots, W_n^{(m)}=k_m\right) \to P\left(J(\tau^{(1)})=k_1, \ldots, J(\tau^{(m)})=k_m\right)$$

as $n \to \infty$, where $\left(J(\tau^{(1)}), \ldots, J(\tau^{(m)})\right)$ is the reverse independent increment Poisson vector with mean $(\tau^{(1)}, \ldots, \tau^{(m)})$. Similarly, if instead, for $0 \leq \tau^{(1)} < \ldots < \tau^{(m)} < \infty$, if the level $u_n^{(\mu)}$ satisfies

$$F\left(u_n^{(\mu)}\right) = \frac{\tau^{(\mu)}}{n} + o\left(\frac{1}{n}\right),$$

$\mu = 1, \ldots, m$, then letting $T_n^{(\mu)} = \sum_{i=1}^{n} I_{[\xi_i \leq u_n^{(\mu)}]}$ we have

$$P\left(T_n^{(1)}=k_1, \ldots, T_n^{(m)}=k_m\right) \to P\left(I(\tau^{(1)})=k_1, \ldots, I(\tau^{(m)})=k_m\right),$$

where $\left(I(\tau^{(1)}), \ldots, I(\tau^{(m)})\right)$ is the independent increment Poisson

vector with mean $(\tau^{(1)}, \ldots, \tau^{(m)})$.

To complete our discussion of limiting joint distributions of fixed rank order statistics from i.i.d. sequences, we illustrate the use of the above representations by computing the asymptotic joint distribution of the maximum $_1\xi^{(n)}$ and the second maximum $_2\xi^{(n)}$, when (1.2.1) holds. Write $G(x) = e^{-\tau(x)}$ and let J be the reverse τ-Poisson process. For $x_1 \le x_2$ such that $G(x_2) > 0$, or equivalently, $\tau(x_2) < \infty$, we have

$$\begin{aligned}\lim_n P\left(\frac{_1\xi^{(n)} - b_n}{a_n} \le x_1, \frac{_2\xi^{(n)} - b_n}{a_n} \le x_2\right) &= P\big(J(x_1)=0, J(x_2)\le 1\big)\\ &= P\big(J(x_1) = 0\big)\\ &= e^{-\tau(x_1)}\\ &= G(x_1),\end{aligned}$$

since $J(x_1) - J(x_2)$ is nonnegative. If $G(x_2) = 0$, then $\tau(x_2) = \infty$ and thus also $\tau(x_1) = \infty$ since τ is nonincreasing, so that $G(x_1) = 0$; hence $J(x_1) = J(x_2) = +\infty$ almost surely and therefore $P\big(J(x_1)=0, J(x_2)\le 1\big) = 0 = G(x_1)$. For $x_1 > x_2$ such that $G(x_2) > 0$ we have

$$\begin{aligned}&\lim_n P\left(\frac{_1\xi^{(n)} - b_n}{a_n} \le x_1, \frac{_2\xi^{(n)} - b_n}{a_n} \le x_2\right)\\ &\quad= P\big(J(x_1) = 0, J(x_2) \le 1\big)\\ &\quad= \sum_{j=0}^{1} P\big(J(x_1) = 0, J(x_2)-J(x_1) = j\big)\\ &\quad= \sum_{j=0}^{1} P\big(J(x_1) = 0\big)P\big(J(x_2)-J(x_1) = j\big)\\ &\quad= e^{-\tau(x_1)}\left\{e^{-(\tau(x_2)-\tau(x_1))}\big(1 + \tau(x_2) - \tau(x_1)\big)\right\}\\ &\quad= e^{-\tau(x_2)}\big(1 + \tau(x_2) - \tau(x_1)\big)\end{aligned}$$

$$= G(x_2)\left[1 + \log \frac{G(x_1)}{G(x_2)}\right],$$

and the same conclusion also holds for $G(x_2) = 0$ if we interpret the last expression to be zero. Thus we have shown that if

$$P\left[\frac{{}_1\xi^{(n)} - b_n}{a_n} \le x_1, \frac{{}_2\xi^{(n)} - b_n}{a_n} \le x_2\right] \to \text{d.f. } H(x_1,x_2)$$

as $n \to \infty$ for all continuity points (x_1,x_2) of H, then H is of the form

$$(1.2.15)\quad H(x_1,x_2) = \begin{cases} 0 & , \quad x_1 > x_2 \text{ and } G(x_2) = 0 \\ G(x_2)\left(1 + \log \frac{G(x_1)}{G(x_2)}\right) & , \quad x_1 > x_2 \text{ and } G(x_2) > 0 \\ G(x_1) & , \quad x_1 \le x_2 , \end{cases}$$

where G is the one-dimensional limiting distribution of the normalized maximum $a_n^{-1}({}_1\xi^{(n)} - b_n)$, which necessarily exists. This agrees with the expression given by Welsch (1972, Lemma 1), who obtained it from alternative calculations. We might mention that this result is also valid if G is degenerate, although it may be seen that in such a case H is also (and conversely, if a two-dimensional law H of the above form is degenerate, then so must be G).

A considerable amount of research has been directed toward the development of a corresponding theory of the asymptotic distributions of extreme order statistics of sequences of r.v.'s which are not necessarily i.i.d. In most instances the identically distributed assumption is retained, and in fact strict stationarity is assumed, and the dependence structures considered involve conditions related to "mixing." One such condition is that of "strong mixing," which we say is satisfied by the sequence ξ_n if

$$(1.2.16) \qquad \sup_{j\ge 1} \sup_{\substack{A\in\mathcal{F}_{1,j} \\ B\in\mathcal{F}_{j+\ell,\infty}}} |P(AB) - P(A)P(B)| \le g(\ell) \downarrow 0 \quad \text{as} \quad \ell \to \infty,$$

for some "mixing" function (sequence) $g(\ell)$, where $\mathcal{F}_{1,j}$ and $\mathcal{F}_{j+\ell,\infty}$ are respectively the σ-fields generated by $\{\xi_1, \ldots, \xi_j\}$ and $\{\xi_{j+\ell}, \xi_{j+\ell+1}, \ldots\}$. As an example we mention that the strong mixing condition is obviously satisfied by m-dependent sequences ξ_n, by which we mean that for each $j \ge 1$, the σ-fields $\mathcal{F}_{1,j}$ and $\mathcal{F}_{j+\ell,\infty}$ are independent, and hence $g(\ell) = 0$, if $\ell > m$. Although the asymptotic theory for extreme order statistics has been considered for other types of dependent sequences, for example Markovian, it is the strong mixing and related dependence conditions to which extensions of the methods and results of the i.i.d. theory have been found to be particularly well suited. Moreover, the case of stationary normal sequences conveniently fits into the general framework, and here the dependence conditions may be formulated in terms of covariances, which as is well known provide for a complete characterization of normal sequences. In the remainder of this section we state some of the results that have been obtained, including those valid under conditions weaker than but related to strong mixing.

Thus suppose ξ_n is a strictly stationary sequence satisfying the strong mixing condition (1.2.16), and let ${}_1\xi^{(n)} = \max\{\xi_1, \ldots, \xi_n\}$. Loynes (1965) has shown that the three limit types result of the i.i.d. case is also valid here; that is, if (1.2.1) holds for some constants $a_n > 0$, b_n, then G is necessarily of one of the three types listed in (1.2.3).

Since, however, the strong mixing condition is rather restrictive and somewhat difficult to verify in practice, it being necessary to

consider all pairs of events in the σ-fields $\mathcal{F}_{1,j}$ and $\mathcal{F}_{j+\ell,\infty}$, and moreover, since not all of the events generated by the r.v.'s ξ_n are actually relevant to the study of the maximum term ${}_1\xi^{(n)}$, one might hope that the strong mixing condition could be weakened while still retaining the three limit types result. This motivation has led to the following condition, proposed by Leadbetter (1974b, 1977). Let u_n be any sequence of real numbers, and for a finite set of integers $i_1, \ldots, i_m$ let $F_{i_1,\ldots,i_m}(u_n) = P(\xi_{i_1} \le u_n, \ldots, \xi_{i_m} \le u_n)$.

$D(u_n)$: $|F_{i_1,\ldots,i_p,j_1,\ldots,j_q}(u_n) - F_{i_1,\ldots,i_p}(u_n) F_{j_1,\ldots,j_q}(u_n)| \le g_{n,\ell}$

for any integers $1 \le i_1 < \ldots < i_p < j_1 < \ldots < j_q \le n$, $j_1 - i_p \ge \ell$, where $g_{n,\ell}$ is nonincreasing in ℓ and

$$\lim_n g_{n,\ell_n} = 0$$

for some sequence $\ell_n \to \infty$ and $\ell_n = o(n)$.

Suppose ξ_n is stationary and that (1.2.1) holds. Leadbetter has shown that if $D(u_n)$ is satisfied for the sequence $u_n = a_n x + b_n$ for all real x, then the limit law G must be of one of the three types (1.2.3). It is clear that the requirement that $D(u_n)$ hold for $u_n = a_n x + b_n$ for each x is indeed a weaker condition than strong mixing, there being fewer pairs of events, and only events of a certain form, which need be considered.

From these results we might then conjecture that, under the same conditions, for other fixed rank terms ${}_k\xi^{(n)}$, and for the joint distributions of two or more fixed rank terms, the possible limit laws are the same as those which can arise under the i.i.d. assumption. However, this is not correct, and thus apparently under the dependence conditions

we are considering the generalization of the i.i.d. three limit types result is peculiar to the (first) maximum. We illustrate with the following simple example given by Welsch (1972) for the one-dimensional case $k = 2$. Let z_n be i.i.d. with marginal d.f. F and suppose for some $a_n > 0$, b_n we have that $F^n(a_n x + b_n) \to G(x)$ for all real x; that is, F belongs to the domain of attraction for maxima of the limit law G. Now define $\xi_n = \max\{z_n, z_{n+1}\}$, $n \geq 1$. Then ξ_n is strictly stationary and satisfies the strong mixing condition (1.2.16), and in fact is 1-dependent, and we have that

$$P\left\{\frac{{}_2\xi^{(n)} - b_n}{a_b} \leq x\right\} \to G(x)$$

for all x. We conclude that the possible limit laws for second maxima include those for first maxima, so that Smirnov's generalization of the three types result no longer is valid. Results further indicating the actual size of the class of possible limit laws for the case $k = 2$, and for the joint laws of first and second maxima, have been obtained by Welsch (1971) and Mori (1976). Presumably more complicated results could as well be obtained for the cases $k > 2$ and for other joint ranks.

Another and related problem of significant interest is to actually find the limiting distributions of extreme order statistics from dependent sequences, and in particular, for what is probably the situation most useful and easiest to handle, to show that the limiting law of a sequence of terms ${}_k\xi^{(n)}$ (normalized) is the same as if the ξ_n were in fact i.i.d., assuming this exists. To deal with the problem, for a given stationary sequence ξ_n we let $\hat{\xi}_n$ be an i.i.d. sequence of r.v.'s with the same marginal d.f. as ξ_n ; $\hat{\xi}_n$ may be called the

"independent sequence associated with ξ_n " (cf. Loynes (1965)). We let ${}_k\hat{\xi}^{(n)}$ denote the k-th largest of $\hat{\xi}_1, \ldots, \hat{\xi}_n$, for each $n \ge k \ge 1$.

As suggested by the 1-dependent example given above, even fairly stringent mixing assumptions would not be sufficient to guarantee that ${}_k\xi^{(n)}$ and ${}_k\hat{\xi}^{(n)}$ (normalized) have the same asymptotic distribution, when these exist, and hence additional assumptions must necessarily be made. For the case $k = 1$ a condition leading to a result of the kind desired has been formulated by O'Brien (1974). Let u_n be a real sequence such that $F(u_n) < 1$ for all n, where F is the marginal d.f. of the stationary sequence ξ_n, assumed to satisfy the strong mixing condition (1.2.16).

R_1: There exist sequences of positive integers p_n, q_n, and r_n such that, as $n \to \infty$, $r_n \to \infty$, $r_n g(q_n) \to 0$, $p_n^{-1} q_n \to 0$, and $r_{n+1} p_{n+1} (r_n p_n)^{-1} \to 1$, and such that

$$\frac{1}{p_n(1-F(u_n))} \sum_{i=1}^{p_n-1} (p_n - 1) P(\xi_1 > u_{t_n}, \xi_{i+1} > u_{t_n}) \to 0,$$

where $t_n = r_n(p_n + q_n)$.

O'Brien has shown that for constants $a_n > 0$, b_n,

$$P\left(\frac{{}_1\hat{\xi}^{(n)} - b_n}{a_n} \le x\right) \to G(x) \quad \text{if and only if} \quad P\left(\frac{{}_1\xi^{(n)} - b_n}{a_n} \le x\right) \to G(x),$$

for $0 < G(x) < 1$, if R_1 holds for the sequence $u_n = a_n x + b_n$. (Loynes (1965) originally showed the "if" part, using $r_n = n$.) Of course, that R_1 should hold for $u_n = a_n x + b_n$ for each x is sufficient for the limit laws of ${}_1\xi^{(n)}$ and ${}_1\hat{\xi}^{(n)}$ to coincide. O'Brien also gives an example for which R_1 is not satisfied for all

$u_n = a_n x + b_n$ and for which the limiting distributions of ${}_1\xi^{(n)}$ and ${}_1\hat{\xi}^{(n)}$ exist but actually differ, though still being of the same type (which would correspond to a change in the normalizing constants a_n, b_n).

The situation for other fixed ranks, and for joint distributions, has also received attention. As a particular example, for the joint distribution of the first and second maxima, Welsch (1971) has shown that under strong mixing and another condition of a type similar to R_1, the joint limit law $H(x_1, x_2)$ is the same as if the ξ_n were i.i.d., that is, of the form given by (1.2.15). Implicit, of course, is the requirement that this (nondegenerate) joint law exist, which as we know is equivalent to the condition that the first maximum (normalized) have a nondegenerate limit law.

The search for conditions weaker than strong mixing to use in establishing that the maximum of a stationary sequence has the same limiting distribution as it would if the sequence were i.i.d., has been motivated by the following result of Berman (1964): If ξ_n is a stationary normal sequence having zero means, unit variances, and covariances $r_n = E\,\xi_1 \xi_{n+1}$, and if either

(1.2.17) $$r_n \log n \to 0 \quad \text{as} \quad n \to \infty$$

or

(1.2.18) $$\sum_{n=1}^{\infty} r_n^2 < \infty,$$

then

$$P\left\{\frac{{}_1\xi^{(n)} - b_n}{a_n} \le x\right\} \to e^{-e^{-x}}$$

for all real x, where a_n, b_n are given by (1.2.6). Since there is no general relationship between strong mixing and (1.2.17) or (1.2.18), except that strong mixing implies (1.2.18) (see Deo (1973a)), the normal case cannot be included in the result of O'Brien and Loynes. However, Leadbetter (1974b) has established a relationship between these covariance conditions for the normal case and conditions applicable to general stationary sequences which lead to the desired conclusions, by again employing $D(u_n)$ and the following condition similar to R_1 :

$$D'(u_n): \quad \overline{\lim_n}\; n \sum_{i=1}^{n-1} P(\xi_1 > u_{nr},\ \xi_{1+i} > u_{nr}) = o\left(\frac{1}{r}\right) \quad \text{as} \quad r \to \infty.$$

Leadbetter has shown that if $\mathcal{L}\left[a_n^{-1}\left({}_1\hat{\xi}^{(n)} - b_n\right)\right] \to G$ and if both $D(u_n)$ and $D'(u_n)$ hold for each $u_n = a_n x + b_n$, then for *every* $k \geq 1$,

$$P\left[\frac{{}_k\xi^{(n)} - b_n}{a_n} \leq x\right] \to G(x) \sum_{s=0}^{k-1} \frac{\left(-\log G(x)\right)^s}{s!}$$

as $n \to \infty$, for all real x, where the right hand side is defined to be zero if $G(x) = 0$. That is, ${}_k\xi^{(n)}$ and ${}_k\hat{\xi}^{(n)}$ have the same asymptotic distribution, and under the same normalization, for all $k \geq 1$. This result for $k > 1$ follows as a corollary to the more general Poisson convergence of the number of exceedances by $\xi_1, \ldots, \xi_n$ of the level u_n satisfying $n\left(1 - F(u_n)\right) \to \tau$, F being the marginal d.f. of ξ_1 , which we have seen to hold in the i.i.d. case, and which Leadbetter has shown to also hold for stationary, dependent sequences under both the conditions $D(u_n)$ and $D'(u_n)$. Moreover, in the normal case either of the covariance conditions (1.2.17) or (1.2.18) imply both $D(u_n)$ and $D'(u_n)$ for $u_n = a_n x + b_n$ for all real x, where

a_n, b_n are given by (1.2.6), a fact which lets us regard the result of Berman as a special case of a result for more general dependent sequences. It is perhaps worth mentioning that it has more recently been established (see Leadbetter, et. al. (1977)) that these results for normal sequences are also valid if the covariances r_n satisfy

$$(1.2.19) \quad r_n \to 0 \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} (r_j)(\log j)e^{\gamma|r_j|\log j} \to 0 \quad \text{as} \quad n \to \infty$$

for some $\gamma > 2$, a condition somewhat weaker than either (1.2.17) or (1.2.18).

Finally in this section we remark that it is of interest to consider other types of situations in which there are limiting distributions of extreme order statistics from stationary, dependent sequences. One such result has been obtained for the maximum term $(k = 1)$ of a stationary normal sequence ξ_n by Mittal and Ylvisaker (1976), who have shown that under a different but equally meaningful covariance condition for which (1.2.17)-(1.2.19) (and hence the strong mixing condition) necessarily fail to hold, the maximum term ${}_1\xi^{(n)}$ is actually asymptotically normal, with normalizing constants a_n, b_n different from those given by (1.2.6). Here it would be interesting to see what are the limiting distributions for other fixed ranks, and for joint ranks, if these exist. Another problem, pertaining to general stationary, dependent sequences, would involve determining if it can happen that the fixed rank terms of a sequence ξ_n have limiting distributions while those of the associated independent sequence do not, that is, the marginal d.f. belongs to no domain of attraction. Of course, for example, for strongly mixing sequences such would require some "large degree" of dependence between the r.v.'s ξ_i and ξ_j when

$|i-j|$ is "small," in order that O'Brien's R_1 condition not hold.

We leave such questions as these open, however, along with potential analagous results in the case of increasing ranks. In fact, when dealing with that case we will only consider the problem of obtaining general dependence conditions insuring that the asymptotic distributions of the increasing rank order statistics of a stationary sequence ξ_n are the same as if the ξ_n were i.i.d., where we assume the latter exists.

1.3. Order Statistics of Increasing Rank.

We now discuss the theory of asymptotic distributions of sequences of order statistics $\xi_{k_n}^{(n)}$ or ${}_{k_n}\xi^{(n)}$, where the ξ_n are i.i.d. and k_n is an increasing rank sequence. Extensive studies of the possible limit laws and their domains of attraction have been made by Smirnov (1952, 1967), Chibisov (1964), Cheng (1965), and Wu (1966). In all cases considered it is assumed that the relative rank sequence $\frac{k_n}{n}$ tends to a limit λ, $0 \leq \lambda \leq 1$, as $n \to \infty$, and the results obtained depend upon the value of λ and, when $0 < \lambda < 1$, upon how rapidly the sequence of relative ranks converges. Since in most instances smallest increasing rank order statistics rather than largest have been considered, we will here primarily deal with sequences $\xi_{k_n}^{(n)}$ of smallest terms.

An important and useful preliminary result, corresponding to (1.2.7) and (1.2.12) for fixed ranks, is the following, which was originally given by Smirnov (1952): Let ξ_n be i.i.d. with marginal d.f. F. Then for given constants $a_n > 0$, b_n, the relation

$$P\left\{\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right\} \to G(x) \tag{1.3.1a}$$

holds if and only if

$$\frac{n}{\sqrt{k_n}} \cdot \frac{F(a_n x + b_n) - \frac{k_n}{n}}{\sqrt{1 - \frac{k_n}{n}}} \to u(x), \tag{1.3.1b}$$

as $n \to \infty$, where $u(x)$ is defined by

$$G(x) = \Phi\{u(x)\},$$

Φ being the standard normal d.f. For a central rank sequence k_n, $\frac{k_n}{n} \to \lambda$, $0 < \lambda < 1$, (1.3.1b) may be written as

$$\sqrt{n}\,\frac{F(a_n x + b_n) - \frac{k_n}{n}}{\sqrt{\lambda(1-\lambda)}} \to u(x), \tag{1.3.2}$$

and for an intermediate rank sequence k_n and for x such that $0 < G(x) < 1$, (1.3.1b) becomes

$$F(a_n x + b_n) = \frac{k_n}{n} + \frac{u(x)\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right). \tag{1.3.3}$$

Similar relations of course hold also for sequences of largest terms ${}_{k_n}\xi^{(n)}$, and by considering the increasing rank sequence k_n' given by $k_n' = n - k_n + 1$, it is easy to show that, as an example, for the particular case that k_n is an intermediate sequence of right ranks, for x such that $0 < G(x) < 1$, we have

$$P\left\{\frac{{}_{k_n}\xi^{(n)} - b_n}{a_n} \le x\right\} \to G(x)$$

if and only if

$$1 - F(a_n x + b_n) = \frac{k_n}{n} - \frac{u(x)\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right), \tag{1.3.4}$$

where $G(x) = \Phi(u(x))$.

As in the fixed ranks case, we may also consider a related problem involving the number of exceedances of an appropriate level by the sequence of r.v.'s ξ_n. In particular, for intermediate rank sequences k_n, let u be a given real number and suppose the level u_n satisfies

$$(1.3.5) \qquad 1 - F(u_n) = \frac{k_n}{n} - \frac{u\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right).$$

Now let $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$ be the number of exceedances of u_n by $\xi_1, \ldots, \xi_n$. Then we have that

$$(1.3.6) \qquad P(W_n < k_n) \to \Phi(u) \quad \text{as} \quad n \to \infty,$$

a result which is clear from the following lemma, which we will also employ for other purposes in Chapter II.

Lemma 1.3.1. If $0 < p_n < 1$ is a sequence satisfying

$$(1.3.7) \qquad p_n = \frac{k_n}{n} - \frac{u\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right)$$

for some real number u, and where $k_n \to \infty$, $k_n = o(n)$, then

$$(1.3.8) \qquad \sum_{r=0}^{k_n} \binom{n}{r} p_n^r (1 - p_n)^{n-r} \to \Phi(u).$$

The result is also valid if the upper limit of summation is $k_n - 1$ instead of k_n.

Proof: This follows from Lemma 1.4.1 of Leadbetter (1977), which is in fact more general, but for completeness we give a proof here. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli variables with mean p_n. Then

$$\sum_{r=0}^{k_n} \binom{n}{r} p_n^r (1 - p_n)^{n-r} = P\Big(\sum_{i=1}^{n} X_i \le k_n\Big)$$

$$(1.3.9) \qquad = P\left(\frac{\sum_{i=1}^{n} X_i - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{k_n - np_n}{\sqrt{np_n(1-p_n)}}\right).$$

From the Berry-Esseen bound (see, for example, van Beek (1972)) we have

$$\left| P\left(\frac{\sum_{i=1}^{n} X_i - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{k_n - np_n}{\sqrt{np_n(1-p_n)}}\right) - \Phi\left(\frac{k_n - np_n}{\sqrt{np_n(1-p_n)}}\right)\right| \le \frac{C}{\sqrt{np_n(1-p_n)}}$$

for some constant C, not depending on n. Now from (1.3.7) the right side tends to zero, while

$$\frac{k_n - np_n}{\sqrt{np_n(1-p_n)}} \to u.$$

This along with (1.3.9) gives (1.3.8).

In a similar way we may consider the number of $\xi_1, \ldots, \xi_n$ which do not exceed a suitably chosen u_n level, which corresponds to the problem of limiting distributions for k_n-th smallest order statistics, again where k_n is an intermediate rank sequence. For a given real number u, we now suppose the level u_n satisfies

$$(1.3.10) \qquad F(u_n) = \frac{k_n}{n} + \frac{u\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right).$$

Then letting $T_n = \sum_{i=1}^{n} I_{[\xi_i \le u_n]}$, we obtain from the above lemma and the trivial relation $\Phi(u) = 1 - \Phi(-u)$,

$$P(T_n \ge k_n) \to \Phi(u).$$

It is also possible to formulate the exceedance problem for central rank sequences k_n, which will not be of primary interest to us

but which we will later have occasion to deal with to some extent. Again let u be a real number and let k_n be a central rank sequence with limiting relative rank λ, $0 < \lambda < 1$. Suppose the level u_n satisfies

$$1 - F(u_n) = \frac{k_n}{n} - \frac{u\sqrt{\lambda(1-\lambda)}}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{1.3.11}$$

and let $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$. Then using the more general form of Lemma 1.3.1 given by Leadbetter (1977), we have that

$$P(W_n < k_n) \to \Phi(u).$$

Let us now return to the problem of limiting distributions of increasing rank order statistics. We will mainly deal with the asymptotic theory for the intermediate ranks case, but for completeness we first briefly consider that for central rank terms. Under the assumption that $\sqrt{n}\left(\frac{k_n}{n} - \lambda\right) \to t$, $0 < \lambda < 1$, $-\infty < t < \infty$, Smirnov (1952) showed that any limit law for $\xi_{k_n}^{(n)}$ (normalized) must be of one of the following four types:

$$G_1(x) = \begin{cases} 0 & , \quad x < 0 \\ \Phi(cx^\alpha) & , \quad x \geq 0 \end{cases} \qquad (c>0,\ \alpha>0)$$

$$G_2(x) = \begin{cases} \Phi(-c|x|^\alpha) & , \quad x < 0 \\ 1 & , \quad x \geq 0 \end{cases} \qquad (c>0,\ \alpha>0)$$

$$G_3(x) = \begin{cases} \Phi(-c_1|x|^\alpha) & , \quad x < 0 \\ \Phi(c_2x^\alpha) & , \quad x > 0 \end{cases} \qquad (c_1>0,\ c_2>0,\ \alpha>0) \tag{1.3.12}$$

$$G_4(x) = \begin{cases} 0 & , \quad x < -1 \\ \tfrac{1}{2} & , \quad -1 \leq x \leq 1 \\ 1 & , \quad x \geq 1\ . \end{cases}$$

Moreover, Smirnov found necessary and sufficient conditions completely specifying the domain of attraction of each of these laws. Other results are valid when the quantity $\sqrt{n}\left(\frac{k_n}{n} - \lambda\right)$ tends to $\pm\infty$ or is bounded but without limit have been obtained by Wu (1966). Here we only provide a well-known particular example of attraction for central order statistics, given by the following result, which Cheng (1965) has proven by using the equivalence of (1.3.1a) and (1.3.1b): For a d.f. F, if there is a number x_λ such that $F(x_\lambda) = \lambda$, $0 < \lambda < 1$, and inside an interval $(x_\lambda-\varepsilon, x_\lambda+\varepsilon)$ F has a derivative F' which is continuous at x_λ, and $F'(x_\lambda) > 0$, then F belongs to the domain of attraction of the standard normal law Φ for any rank sequence k_n such that $\frac{k_n}{n} \to \lambda$; specifically,

(1.3.13) $$P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) \to \Phi(x)$$

for all real x; where a_n, b_n are given by

$$F(b_n) = \frac{k_n}{n}$$

(1.3.14) and

$$a_n = \frac{\sqrt{k_n}}{nF'(b_n)}\sqrt{1-\lambda}\ .$$

Under the additional assumption that $\sqrt{n}\left(\frac{k_n}{n} - \lambda\right) \to 0$, by employing an appropriate change in the normalizing constants, according to the usual criteria (see Feller (1966, p. 246)), we find that (1.3.13)-(1.3.14) becomes the statement that $\sqrt{n}\left(\xi_{k_n}^{(n)} - x_\lambda\right)$ is asymptotically normal with mean zero and variance $\lambda(1-\lambda)[F'(x_\lambda)]^{-2}$, where x_λ is the unique λ-quantile of F. This result has been given by a number of authors, including, for example, Wilks (1962, p. 273).

Results answering the question of the possible limit laws and their domains of attraction for intermediate order statistics of i.i.d. sequences have been obtained by Chibisov (1964), Cheng (1965), and Wu (1966). Chibisov has shown that for a particular class of rank sequences satisfying

$$\frac{k_n}{n^{\theta}} \to \ell^2 , \tag{1.3.15}$$

where $\ell > 0$ and $0 < \theta < 1$, the possible limit laws of sequences $\xi^{(n)}_{k_n}$ are given by

$$G_1(x) = \begin{cases} 0 & , \quad x \le 0 \\ \Phi(a \log x) & , \quad x > 0 \end{cases} \qquad (a > 0)$$

$$G_2(x) = \begin{cases} \phi(-a \log x) & , \quad x < 0 \\ 1 & , \quad x \ge 0 \end{cases} \qquad (a > 0) \tag{1.3.16}$$

$$G_3(x) = \Phi(x) \qquad , \quad -\infty < x < \infty .$$

We note that each of these limit laws is continuous, so that (1.3.1a) would necessarily hold for all real x. Moreover, as is evident upon replacing $G(x)$ by $1 - G(-x)$, for the same class of intermediate rank sequences, these are also the possible limit laws for k_n-th largest terms ${}_{k_n}\xi^{(n)}$. The following theorem of Wu (1966) extends these results to more general intermediate rank sequences, subject only to a relatively mild restriction (which in fact can be weakened).

Theorem 1.3.2. Suppose ξ_n is i.i.d. and let k_n be a monotonically increasing (not necessarily strictly) intermediate rank sequence. If there are constants $a_n > 0$, b_n such that

$$P\left\{\frac{\xi^{(n)}_{k_n} - b_n}{a_n} \le x\right\} \to \text{d.f. } G(x)$$

for all continuity points x of G, then G is of one of the three types listed in (1.3.16).

For the intermediate rank sequences he considered, Chibisov (1964) found necessary and sufficient conditions characterizing the domains of attraction of each of the limit laws (1.3.16). As in the fixed ranks case, membership of a distribution in a domain of attraction for smallest terms is determined by its behavior near its left endpoint. However, the domains of attraction are highly dependent upon the particular rank sequence chosen, even to the extent that for each of the laws G_1 and G_2 in (1.3.16), there is no distribution attracted under each of two rank sequences satisfying (1.3.15) for different pairs (θ,ℓ). In addition, there are rank sequences not satisfying (1.3.15) for which only the normal law $G_3 = \Phi$ is possible (cf. Wu (1966, p. 757)). Thus at present there are no general characterizations, valid for all intermediate rank sequences k_n, of the domains of attraction of the possible limit laws.

However, several particular results dealing with domains of attraction are known and cover many situations of interest. The following two theorems due to Cheng (1965) show that the normal law has a nonvoid domain of attraction for every intermediate rank sequence, and moreover, there are distributions attracted to the normal law for every intermediate rank sequence. The first of these is fundamental to the results we will develop in Chapter IV.

<u>Theorem 1.3.3</u>. Let ξ_n be an i.i.d. sequence with marginal d.f. F. Suppose that F has a finite left endpoint x_0 such that $F(x_0) = 0$ (that is, there is no probability mass at x_0), and in an interval

$(x_0, x_0+\delta)$, the derivative F' exists and is positive, and such that $\lim_{x \downarrow x_0} F'(x) > 0$. Then for each intermediate rank sequence k_n, we have

$$P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) \to \Phi(x)$$

as $n \to \infty$, for all real x, where $a_n > 0$, b_n (depending on k_n) are given by

$$F(b_n) = \frac{k_n}{n},$$

(1.3.17) and

$$a_n = \frac{\sqrt{k_n}}{nF'(b_n)}.$$

Note that under the implicit assumption that $c \equiv \lim_{x \downarrow x_0} F'(x)$ exists, we may replace a_n by

(1.3.18) $$a_n' = \frac{\sqrt{k_n}}{nc}.$$

Theorem 1.3.4. Suppose ξ_n is i.i.d. with marginal d.f. F. Let $M > 0$, $p \le 1$, and d be given real numbers. If in an interval $(-\infty, d)$ F is twice differentiable and such that in this interval, $F'(x) > 0$,

$$\left|\frac{F(x)}{x^p F'(x)}\right| < M,$$

and

$$\left|\frac{F(x)F''(x+o(x^p))}{[F'(x)]^2}\right| < M,$$

then for each intermediate rank sequence k_n,

$$P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) \to \Phi(x)$$

as $n \to \infty$, for all real x, where

$$F(b_n) = \frac{k_n}{n}$$

and

$$a_n = \frac{\sqrt{k_n}}{nF'(b_n)} .$$

As an example, Cheng shows that the standard normal distribution Φ satisfies the conditions of Theorem 1.3.4.

These results can of course be rephrased to give sufficient conditions involving a corresponding behavior of the marginal d.f. F near its finite or infinite right endpoint, for the sequence ${}_{k_n}\xi^{(n)}$ of largest rank terms to have a limiting normal distribution, that is,

$$P\left(\frac{{}_{k_n}\xi^{(n)} - b_n}{a_n} \le x\right) \to \Phi(x)$$

for all real x. In this situation, when such conditions are satisfied, the appropriate normalizing constants are given by

$$F(b_n) = 1 - \frac{k_n}{n}$$

(1.3.19) and

$$a_n = \frac{\sqrt{k_n}}{nF'(b_n)} .$$

As is clear from symmetry, the normal distribution would again be an example, this time of a distribution attracted to itself for any intermediate sequence of right ranks.

Finally we are led to consider the class of distributions attracted to the normal law for every intermediate rank sequence. For smallest terms ${}_{k_n}\xi^{(n)}$ Smirnov (1967) has shown that this class consists of all distributions F satisfying the following condition: There exists a continuous function $Q(z) \to 0$ as $z \downarrow x_0$, such that for all real x,

$$\frac{F\left[z\left(1+xQ(z)\right)\right]-F(z)}{F\left[z\left(1+Q(z)\right)\right]-F(z)} \to x, \quad \text{as} \quad z \downarrow x_0 ,$$

where x_0 $(\geq -\infty)$ is the left endpoint of F. This condition is analagous to Gnedenko's (1943) characterization of the domain of attraction for maxima of the law $G_3(x) = e^{-e^{-x}}$, in that some suitable continuous function is to be found. However, as earlier noted, alternative conditions have been obtained for that case, and it would be of interest to see if other conditions could as well be found for the intermediate case.

CHAPTER II

LIMITING DISTRIBUTIONS OF INTERMEDIATE ORDER STATISTICS FROM STATIONARY SEQUENCES

2.1. Preliminaries.

In this chapter we study the problem of finding sufficient conditions under which we can obtain the asymptotic distributions of increasing rank order statistics from stationary, dependent sequences. In particular we consider the situation in which, for a given dependent sequence ξ_n, and increasing rank sequence k_n, the r.v. $\xi_{k_n}^{(n)}$ or ${}_{k_n}\xi^{(n)}$ (normalized) has a limiting distribution which is the same as if the ξ_n were i.i.d., where of course we assume that the marginal d.f. of ξ_1 belongs to the domain of attraction of a limit law for the sequence of ranks k_n. The conditions we impose are of a type similar to those which we have examined in the fixed ranks case, one significant difference being that certain rapid "mixing" rates are assumed. As we might expect, the procedures used in dealing with the same problem for fixed ranks are most readily adapted to handle the asymptotic distribution problem for intermediate order statistics, and the results we establish will be valid only for this case. We have not yet obtained analagous results for central ranks.

Because of the nature of our conditions, and for simplicity in comparing them with those used in the fixed ranks case, especially Leadbetter's (1974b, 1977) $D(u_n)$ and $D'(u_n)$, we will develop and

initially express our results for largest rather than smallest rank intermediate order statistics. Moreover, in our procedures we will in fact deal with the more general problem concerning numbers of high level exceedances by stationary, dependent sequences, as outlined for i.i.d. sequences in the previous chapter. This problem is interesting in its own right and, of course, is intimately related to that of limiting distributions of order statistics of largest rank. However, we will also rephrase our conditions and results as they would pertain to smallest rank terms.

To begin the development let us suppose that ξ_n is a stationary sequence of r.v.'s having marginal d.f. F, that k_n is an intermediate rank sequence, and $_{k_n}\xi^{(n)}$ is the k_n-th largest of $\xi_1, \ldots, \xi_n$. Let u be a given real number, and suppose that we can choose a sequence of levels u_n to satisfy (1.3.5), that is, upon relabeling,

$$1 - F(u_n) = \frac{k_n}{n} - \frac{u\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right) . \tag{2.1.1}$$

According to (1.3.4) this can certainly be done, for any real number u, when F belongs to the domain of attraction of some limit law G, of one of the types listed in (1.3.14), for the sequence k_n. Now let W_n be the number of exceedances of u_n by $\xi_1, \ldots, \xi_n$, that is, $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$. Our objective, then, is to establish that under certain dependence conditions we have that, as in the independent case,

$$P(W_n < k_n) \to \Phi(u), \quad \text{as} \quad n \to \infty. \tag{2.1.2}$$

We have been able to accomplish that by means of each of two somewhat different procedures. In Section 2.3 we give conditions for general intermediate rank sequences under which the desired result

follows immediately from the asymptotic normality of a standardized form of W_n, which we obtain by employing known central limit theory for row sums of triangular arrays of r.v.'s dependent in each row. In the next section, however, we give a procedure for obtaining (2.1.2) by actually approximating the value of $P(W_n < k_n)$ by alternative quantities which can be more easily calculated; the nature of this procedure, however, is such that it is apparently successful only if k_n does not tend to infinity too rapidly, in a sense which we make precise below.

For the given intermediate rank sequence k_n let us define $\theta = \theta(\{k_n\})$ by

$$\theta = \inf\{\theta_1 : k_n = O(n^{\theta_1})\}. \tag{2.1.3}$$

Clearly we have $0 \leq \theta \leq 1$, and it will be convenient to regard θ as a measure of how rapidly k_n tends to infinity. Moreover, regardless of the value of θ, it is evident that we have

$$k_n = o(n^{\theta+\varepsilon}), \quad \text{for every} \quad \varepsilon > 0. \tag{2.1.4}$$

In perhaps the most typical cases we actually have $k_n \sim cn^{\theta}$ for some $c > 0$ and $0 < \theta < 1$, but it is clearly also possible to have $\theta = 0$, indicating a slowly increasing intermediate rank sequence, as for example $k_n \sim c \log n$, or $\theta = 1$, indicating a rapidly increasing intermediate rank sequence, which we would have if, for example, $k_n \sim cn(\log n)^{-1}$.

Our basic strategy is to employ a technique used in various forms by a number of previous authors when dealing with dependent r.v.'s satisfying conditions related to mixing, including its original use in considering central limit theory (see, for example, Hoeffding and

Robbins (1948)). We first let α_n and β_n be sequences of positive integers such that

(2.1.5) $\alpha_n \to \infty$, $\beta_n \to \infty$, $\alpha_n = o(n)$, and $\beta_n = o(\alpha_n)$.

Then, for each large n an integer ℓ_n is determined by the relation

(2.1.6) $n = (\alpha_n + \beta_n)\ell_n + R_n$, $0 \le R_n < \alpha_n + 2\beta_n$;

it is readily checked that

$$\ell_n \sim n\alpha_n^{-1} . \tag{2.1.7}$$

We now partition the first n positive integers into "intervals" $I_1^{(n)}, I_2^{(n)}, \ldots, I_{2\ell_n-1}^{(n)}, I_{2\ell_n}^{(n)}$, of respective alternating "lengths" $\alpha_n, \beta_n, \alpha_n, \beta_n, \ldots, \alpha_n, R_n$. By interval we mean a finite set of consecutive integers, and the length of any interval is simply the number of integers it contains. For any set of integers I, which may not necessarily be an interval, we let $W_I = W_I^{(n)}$ be the number of exceedances of the level u_n by $\{\xi_i: i \in I\}$. (For notational simplicity we will delete explicit dependence upon n .) Then we may write

$$W_n = \sum_{i=1}^{2\ell_n} W_{I_i^{(n)}} .$$

The idea now is to show that, for appropriate choices of α_n and β_n, the r.v.'s $W_{I_i^{(n)}}$ with i even make a negligible contribution asymptotically, in determining the quantity $P(W_n < k_n)$ or the limiting distribution of W_n (standardized). Further, we can deal simultaneously with the $W_{I_i^{(n)}}$, i odd, without too much difficulty, under assumptions analagous to those of the fixed ranks case, and moreover

these latter r.v.'s will be "approximately independent" for large n.

We have not made a significant attempt to judge the relative merits of the two procedures to be presented, other than to note that the one given in Section 2.3 is substantially shorter. One obstacle to making comparisons arises from having to assume various relationships among the sequences α_n, β_n, and k_n. Moreover, in Section 2.4 we will consider stationary normal sequences, giving covariance conditions sufficient for obtaining the results from both procedures, and we will see that in that case the relative strengths of the results "overlap," one admitting weaker covariance conditions for small $\theta = \theta(\{k_n\})$ and the other for large θ.

We now consider our first procedure, involving a direct calculation of the quantity $P(W_n < k_n)$.

2.2. Calculating $P(W_n < k_n)$ When θ Is "Small."

We continue to assume that ξ_n is stationary with marginal d.f. F and k_n is an intermediate rank sequence. We will however in this section make a restriction upon how rapidly k_n tends to infinity.

For the given sequence k_n let s_n be a sequence of positive integers such that

$$s_n \to \infty, \quad s_n = o(k_n^{1/2}). \tag{2.2.1}$$

Let us now suppose that α_n and β_n are sequences of positive integers satisfying (2.1.5), and moreover such that

$$\begin{aligned} &\beta_n k_n = o(\alpha_n s_n), \\ &\alpha_n k_n^2 = o(n), \\ \text{and} \quad &\beta_n k_n^{1/2} = o(\alpha_n). \end{aligned} \tag{2.2.2}$$

In order that these relationships, which will be crucial in the development, be satisfied, it is necessary to assume that k_n not increase too rapidly. In fact, since

$$\frac{k_n^5}{n^2} = \frac{\beta_n k_n}{\alpha_n s_n} \cdot \left(\frac{\alpha_n k_n^2}{n}\right)^2 \cdot \frac{\beta_n k_n^{\frac{1}{2}}}{\alpha_n} \cdot \frac{s_n}{k_n^{\frac{1}{2}}} \cdot \frac{1}{\beta_n^2} \to 0,$$

we see that we are restricting our consideration to rank sequences k_n for which $k_n = o(n^{2/5})$. Moreover, if $0 < \theta = \theta(\{k_n\}) < \frac{2}{5}$, we may actually choose $\alpha_n \sim n^{\lambda}$ and $\beta_n \sim n^{\mu}$ for some $0 < \mu < \lambda < 1$. Then taking $s_n \sim n^{\nu}$, $0 < \nu < \frac{\theta}{2}$, we may restate (2.2.2) as

$$\mu + \theta < \lambda + \nu,$$

$$\lambda + 2\theta < 1,$$

(2.2.3) and

$$\mu + \frac{\theta}{2} < \lambda,$$

which is in fact slightly more restrictive than (2.2.2). Also, if $\theta = 0$, we can satisfy (2.2.2) by taking $\alpha_n \sim n^{\lambda}$ and $\beta_n \sim n^{\mu}$ for any choices of λ and μ such that $0 < \mu < \lambda < 1$.

Now, for a given large n, let $\underline{i}_r = \{i_1, \ldots, i_r\}$, where $i_1 < \ldots < i_r$, be a set of r integers chosen from among the first ℓ_n odd positive integers, ℓ_n being defined by (2.1.6), and where $0 \le r \le k_n - s_n$. (We define $\underline{i}_o = \phi$, and from the relations $\alpha_n k_n = o(n)$ and $\ell_n \sim n\alpha_n^{-1}$ we may assume that $k_n < \ell_n$.) We then define events

$$A_n = \bigcup_{r=0}^{k_n - s_n} \bigcap_{\text{all } \underline{i}_r} \left\{ \left[\bigcup_{i \in \underline{i}_r} \left(W_{I_i^{(n)}} = 1\right) \right] \cap \left[\bigcap_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} \left(W_{I_{i'}^{(n)}} = 0\right) \right] \right\}$$

and

$$B_n = \bigcup_{r=k_n-s_n+1}^{k_n-1} \bigcup_{\text{all } \underline{i}_r} \left\{ \left[\bigcap_{i \in \underline{i}_r} \left(W_{I_i^{(n)}} = 1 \right) \right] \cap \left[\bigcap_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} \left(W_{I_{i'}^{(n)}} = 0 \right) \right] \right\}$$

Thus the event A_n contains all outcomes for which in each of up to $k_n - s_n$ of the odd-indexed intervals $I_1^{(n)}, I_3^{(n)}, \ldots, I_{2\ell_n-1}^{(n)}$ there is exactly one exceedance of the level u_n, given by (2.1.1), by $\xi_1, \ldots, \xi_n$, and in the remaining odd-indexed intervals there are no exceedances; a similar statement holds for the event B_n. We show by means of the following lemmas that the value of $P(W_n < k_n)$ can be approximated by $P(A_n)$.

<u>Lemma 2.2.1.</u> $P(A_n) - P\left(W_{\bigcup_{i \text{ even}} I_i^{(n)}} > s_n \right) \leq P(W_n < k_n)$

$$\leq P(A_n) + P(B_n) + P\left(\bigcup_{i \text{ odd}} W_{I_i^{(n)}} > 1 \right).$$

<u>Proof</u>: The second equality follows from the set relations

$$(W_n < k_n) \subseteq \left(W_{\bigcup_{i \text{ odd}} I_i^{(n)}} < k_n \right) = \left(W_{\bigcup_{i \text{ odd}} I_i^{(n)}} \leq k_n - s_n \right)$$

$$\cup \left(k_n - s_n < W_{\bigcup_{i \text{ odd}} I_i^{(n)}} < k_n \right)$$

$$\subseteq A_n \cup B_n \cup \left(\bigcup_{i \text{ odd}} W_{I_i^{(n)}} > 1 \right).$$

The first inequality is immediate, since

$$(W_n < k_n) \supseteq A_n \cap \left(W_{\bigcup_{i \text{ even}} I_i^{(n)}} < s_n \right)$$

$$= A_n \setminus \left[\left(W_{\bigcup_{i \text{ even}} I_i^{(n)}} \geq s_n \right) \cap A_n \right].$$

<u>Lemma 2.2.2</u>. If the sequences α_n, β_n, and s_n are such that (2.2.2) holds, then

$$P\left(W_{\bigcup_{i \text{ even}} I_i^{(n)}} \geq s_n \right) \to 0, \quad \text{as} \quad n \to \infty.$$

<u>Proof</u>: Using the Markov inequality we have

$$P\left(W_{\bigcup_{i \text{ even}} I_i^{(n)}} \geq s_n \right) \leq s_n^{-1} \, E\, W_{\bigcup_{i \text{ even}} I_i^{(n)}}$$

$$\leq \frac{(\ell_n+1)\beta_n + \alpha_n}{s_n} \cdot P(\xi_1 > u_n),$$

which tends to zero, since from (2.1.1) we have $P(\xi_1 > u_n) \sim \frac{k_n}{n}$, and since $\ell_n \sim \frac{n}{\alpha_n}$.

<u>Lemma 2.2.3</u>. If $n \sum_{j=1}^{\alpha_n - 1} P(\xi_1 > u_n, \xi_{1+j} > u_n) \to 0$, then

$$P\left(\bigcup_{i \text{ odd}} \left(W_{I_i^{(n)}} > 1 \right) \right) \to 0, \quad \text{as} \quad n \to \infty.$$

<u>Proof</u>: To prove this we note that

$$P\left(\bigcup_{i \text{ odd}} \left(W_{I_i^{(n)}} > 1 \right) \right) \leq \ell_n P\left(W_{I_1^{(n)}} > 1 \right)$$

$$\leq \ell_n \sum_{1 \leq i < j \leq \alpha_n} P(\xi_i > u_n, \xi_j > u_n)$$

$$\leq \ell_n \alpha_n \sum_{j=1}^{\alpha_n - 1} P(\xi_1 > u_n, \xi_{1+j} > u_n)$$

by stationarity, and the conclusion follows, since $\ell_n \alpha_n \sim n$.

On combining these lemmas we obtain the following result.

Lemma 2.2.4. If the sequences α_n, β_n, and s_n satisfy the relationships (2.2.2), and if $n \sum_{j=1}^{\alpha_n - 1} P(\xi_1 > u_n, \xi_{1+j} > u_n) \to 0$, then

$$|P(W_n < k_n) - P(A_n)| \leq P(B_n) + o(1),$$

as $n \to \infty$.

Having this preliminary result, we now make the following assumption:

A_1: For the given sequence of levels u_n satisfying (2.1.1) for some real number u, there is a system of sequences $\{\alpha_n, \beta_n, s_n;\ n \geq 1\}$ subject to (2.1.5), (2.2.1), and (2.2.2), and such that the following are satisfied:

$$n \sum_{j=1}^{\alpha_n - 1} P(\xi_1 > u_n, \xi_{1+j} > u_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{2.2.4}$$

and

For each large n, for $p, q \geq 1$ and integers $1 \leq i_1 < \dots < i_p < j_1 < \dots < j_q \leq n$ such that $j_1 - i_p \geq \ell$, we have

$$\begin{aligned} &|P(\xi_{i_1} \leq u_n, \dots, \xi_{i_p} \leq u_n, \xi_{j_1} \leq u_n, \dots, \xi_{j_q} \leq u_n) \\ &\quad - P(\xi_{i_1} \leq u_n, \dots, \xi_{i_p} \leq u_n) \cdot P(\xi_{j_1} \leq u_n, \dots, \xi_{j_q} \leq u_n)| \\ &\quad \leq g_{n,\ell}, \end{aligned} \tag{2.2.5}$$

where

$$g_{n,\beta_n} = o\left(e^{-n^{\kappa}}\right) \quad \text{as} \quad n \to \infty, \tag{2.2.6}$$

for some $\kappa > 1-\lambda > \theta$, where $\alpha_n \sim n^{\lambda}$ and θ is defined by (2.1.3).

We see that (2.2.4) is simply the hypothesis of Lemmas 2.2.3 and 2.2.4 and is a variant of the condition $D'(u_n)$ used in the fixed ranks case. Also we note that (2.2.5)-(2.2.6) is in fact the $D(u_n)$ condition with a certain fast rate of decrease to zero by $g_{n,\ell}$. It will later become clear why we assume such a rate, there being in the increasing ranks case a need to consider events other than just those relevant to fixed ranks, that is, of the form appearing in (2.2.5).

It is also of particular interest here to see exactly how the condition (2.2.5) relates to the strong mixing condition (1.2.16). As is apparent (2.2.5) will be satisfied if (1.2.16) holds with $g(\ell) \downarrow 0$ sufficiently fast. More precisely, suppose that a system $\{\alpha_n, \beta_n, s_n;\ n \geq 1\}$ with $\alpha_n \sim n^\lambda$ and $\beta_n \sim n^\mu$ is chosen subject to (2.1.5), (2.2.1), and (2.2.2), and that (2.2.4) is satisfied. If (1.2.16) holds, then clearly so does (2.2.5) with $g_{n,\ell} = g(\ell)$, and if in addition, $g(\ell) = o(e^{-\ell^{\kappa_1}})$ with $\kappa_1 > \frac{1-\lambda}{\mu}$, then

$$\begin{aligned} g_{n,\beta_n} &= g(\beta_n) \\ &= o(e^{-\beta_n^{\kappa_1}}) \\ &= o(e^{-(1-\varepsilon)n^{\kappa_1\mu}}) \end{aligned}$$

for some $\varepsilon > 0$, and then (2.2.6) will hold for any κ such that $\kappa_1\mu > \kappa > 1-\lambda$. Our condition (2.2.5)-(2.2.6) is of course satisfied by i.i.d. sequences ξ_n, but it is not obvious to what extent we are restricting the class of stationary sequences being considered. However, it should be noted that this "distributional mixing" is potentially much weaker than strong mixing with the sequence $g(\ell)$ tending

to zero at the rate indicated above.

We now present the main result.

Theorem 2.2.5. Let ξ_n be a stationary sequence of r.v.'s with marginal d.f. F, and k_n an intermediate rank sequence such that $0 \le \theta < \frac{2}{5}$. Let the level u_n be defined by (2.1.1) and let $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$ be the number of exceedances of u_n by $\xi_1, \ldots, \xi_n$. If the assumption A_1 holds, then $P(W_n < k_n) \to \Phi(u)$, as $n \to \infty$.

In order to prove this theorem we will show that under A_1,

$$(2.2.7) \qquad P(A_n) \to \Phi(u)$$

and

$$(2.2.8) \qquad P(B_n) \to 0,$$

as $n \to \infty$. The theorem will then follow by Lemma 2.2.4. Throughout, we assume that a system $\{\alpha_n, \beta_n, s_n;\ n \ge 1\}$ for which all of the given conditions are satisfied has been chosen. We begin with the following lemmas.

Lemma 2.2.6. As $n \to \infty$,

$$\sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} \left[P\left(W_{I_1(n)} \ge 1\right) \right]^r \left[P\left(W_{I_1(n)} = 0\right) \right]^{\ell_n - r} \to \Phi(u).$$

Again note that $k_n = o(\ell_n)$ by (2.1.7) and (2.2.2), so that $k_n < \ell_n$ for all large n.

Proof: By Lemma 1.3.1 it suffices to show that

$$P\left(W_{I_1^{(n)}} \geq 1\right) = \frac{k_n - s_n}{\ell_n} - \frac{u\sqrt{k_n - s_n}}{\ell_n} + o\left(\frac{\sqrt{k_n - s_n}}{\ell_n}\right), \tag{2.2.9}$$

as $n \to \infty$. To obtain this we first note that for any nonnegative integer-valued r.v. Z we have the easily proven inequalities $E\,Z \leq P(Z=1) + E\,Z(Z-1) \leq P(Z\geq 1) + E\,Z(Z-1)$. Then writing $Z = W_{I_1^{(n)}} = \sum_{j=1}^{\alpha_n} I_{[\xi_j > u_n]}$ we have

$$\begin{aligned}
0 \leq E\,W_{I_1^{(n)}}\left(W_{I_1^{(n)}} - 1\right) &= \sum_{i,j=1}^{\alpha_n} P(\xi_i > u_n,\ \xi_j > u_n) - \sum_{j=1}^{\alpha_n} P(\xi_j > u_n) \\
&= \sum_{i \neq j} P(\xi_i > u_n,\ \xi_j > u_n) \\
&\leq 2\,\alpha_n \sum_{j=1}^{\alpha_n - 1} P(\xi_1 > u_n,\ \xi_{1+j} > u_n) \\
&= o\left(\frac{1}{\ell_n}\right),
\end{aligned} \tag{2.2.10}$$

by (2.2.4). Hence

$$0 \leq E\,W_{I_1^{(n)}} - P\left(W_{I_1^{(n)}} \geq 1\right) \leq E\,W_{I_1^{(n)}}\left(W_{I_1^{(n)}} - 1\right) = o\left(\frac{1}{\ell_n}\right),$$

and so

$$\begin{aligned}
P\left(W_{I_1^{(n)}} \geq 1\right) &= E\,W_{I_1^{(n)}} - o\left(\frac{1}{\ell_n}\right) \\
&= \alpha_n P(\xi_1 > u_n) - o\left(\frac{1}{\ell_n}\right) \\
&= \frac{\alpha_n k_n}{n} - \frac{u\alpha_n\sqrt{k_n}}{n} + o\left(\frac{\alpha_n\sqrt{k_n}}{n}\right).
\end{aligned} \tag{2.2.11}$$

Now since

$$\begin{aligned}
k_n\left(\frac{1}{\ell_n} - \frac{\alpha_n}{n}\right)\frac{\ell_n}{\sqrt{k_n}} &= \sqrt{k_n}\left(1 - \frac{\alpha_n \ell_n}{n}\right) \\
&= \sqrt{k_n}\left(\frac{n - \alpha_n\ell_n - \beta_n\ell_n}{n} + \frac{\beta_n\ell_n}{n}\right) \\
&< \sqrt{k_n}\left(\frac{\alpha_n + 2\beta_n}{n} + \frac{\beta_n\ell_n}{n}\right)
\end{aligned}$$

$$< \frac{(\alpha_n+2\beta_n)k_n}{n} + O\left(\frac{\beta_n\sqrt{k_n}}{\alpha_n}\right)$$

$$\to 0$$

by (2.2.2), and since $s_n = o(k_n)$, we have

$$\frac{\alpha_n k_n}{n} - \frac{k_n}{\ell_n} = o\left(\frac{\sqrt{k_n}}{\ell_n}\right) = o\left(\frac{\sqrt{k_n - s_n}}{\ell_n}\right) .$$

By inserting this into (2.2.11) along with the relation

$$u\sqrt{k_n} = u\sqrt{k_n - s_n}\,\big(1 + o(1)\big) = u\sqrt{k_n - s_n} + o(\sqrt{k_n - s_n})$$

we obtain (2.2.9).

Lemma 2.2.7.

$$\sum_{r=k_n-s_n+1}^{k_n-1} \binom{\ell_n}{r}\left[P\left(W_{I_1^{(n)}} \ge 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r} \to 0, \quad \text{as} \quad n \to \infty.$$

Proof: To prove this we will use the Berry-Esseen bound (van Beek (1972)): If $X_1, \ldots, X_n$ are i.i.d. r.v.'s each having zero mean, variance σ^2, and absolute third moment $\rho > 0$, then

$$\left| P\left(\frac{X_1 + \ldots + X_n}{\sigma\sqrt{n}} \le x\right) - \Phi(x) \right| \le \frac{C\rho}{\sigma^3\sqrt{n}}$$

uniformly for all real x, where $C < 1$ is an absolute constant.

Now, for each n let $X_{n,1}, \ldots, X_{n,\ell_n}$ be ℓ_n i.i.d. Bernoulli r.v.'s with mean $p_n = P\left(W_{I_1^{(n)}} \ge 1\right)$. Then from the above we have

$$\sum_{r=k_n-s_n+1}^{k_n-1} \binom{\ell_n}{r}\left[P\left(W_{I_1}(n) \geq 1\right)\right]^r \left[P\left(W_{I_1}(n) = 0\right)\right]^{\ell_n - r}$$

$$= P(X_{n,1}+\ldots+X_{n,\ell_n} \leq k_n - 1)$$

$$- P(X_{n,1}+\ldots+X_{n,\ell_n} \leq k_n - s_n)$$

$$\leq \left| P\left(\frac{X_{n,1}+\ldots+X_{n,\ell_n} - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}} \leq \frac{k_n - 1 - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right) - \Phi\left(\frac{k_n - 1 - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right)\right|$$

$$+ \left| P\left(\frac{X_{n,1}+\ldots+X_{n,\ell_n} - \ell_n p_n}{(\ell_n p_n(1-p))^{\frac{1}{2}}} \leq \frac{k_n - s_n - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right) - \Phi\left(\frac{k_n - s_n - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right)\right|$$

$$+ \Phi\left(\frac{k_n - 1 - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right) - \Phi\left(\frac{k_n - s_n - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right)$$

$$\leq \frac{2Kp_n}{\ell_n^{\frac{1}{2}} p_n^{3/2}(1-p_n)^{3/2}} + \Phi\left(\frac{k_n - 1 - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right)$$

$$- \Phi\left(\frac{k_n - s_n - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}}\right),$$

since $\rho_n = E|X_{n,1} - E\,X_{n,1}|^3 = (1 - p_n)^3 p_n + p_n^3(1 - p_n) \leq K\,p_n$ for some constant K not depending on n. Now since $p_n \to 0$ and $\ell_n p_n \to \infty$, by (2.2.9), the first term in the last expression above tends to zero. Also, since $\ell_n p_n(1 - p_n) \sim k_n$ and $s_n = o(k_n^{\frac{1}{2}})$, it only remains to show that

$$\frac{k_n - \ell_n p_n}{(\ell_n p_n(1-p_n))^{\frac{1}{2}}} \tag{2.2.12}$$

converges, as $n \to \infty$. But it follows from (2.2.9) that

$p_n = \frac{k_n}{\ell_n} - \frac{u\sqrt{k_n}}{\ell_n} + o\left(\frac{\sqrt{k_n}}{\ell_n}\right)$, so that clearly we have $k_n - \ell_n p_n \sim u\sqrt{k_n}$,

and again since $\ell_n p_n(1 - p_n) \sim k_n$, the quantity (2.2.12) tends to the limit u.

In the next result we show that $P\left(W_{I_1^{(n)}} \geq 1\right)$ may be replaced by $P\left(W_{I_1^{(n)}} = 1\right)$ in the above.

Lemma 2.2.8.

(a) $$\sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} \left[P\left(W_{I_1^{(n)}} \geq 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r}$$

$$- \sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} \left[P\left(W_{I_1^{(n)}} = 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r} \to 0,$$

as $n \to \infty$.

(b) $$\sum_{r=k_n - s_n + 1}^{k_n - 1} \binom{\ell_n}{r} \left[P\left(W_{I_1^{(n)}} \geq 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r}$$

$$- \sum_{r=k_n - s_n + 1}^{k_n - 1} \binom{\ell_n}{r} \left[P\left(W_{I_1^{(n)}} = 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r} \to 0,$$

as $n \to \infty$.

Proof: The proofs of these two statements are virtually identical, so we only consider (a). We first note the fact that

$$0 \le P\left(W_{I_1}(n) > 1\right) \le E\, W_{I_1}(n)\left(W_{I_1}(n) - 1\right),$$

so that by (2.2.10), $P\left(W_{I_1}(n) > 1\right) = o\left(\frac{1}{\ell_n}\right)$. Thus we have

$$\sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} \left[P\left(W_{I_1}(n) \ge 1\right)\right]^r \left[P\left(W_{I_1}(n) = 0\right)\right]^{\ell_n - r}$$

$$- \sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} \left[P\left(W_{I_1}(n) = 1\right)\right]^r \left[P\left(W_{I_1}(n) = 0\right)\right]^{\ell_n - r}$$

$$= \sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} \left\{\left[P\left(W_{I_1}(n) \ge 1\right)\right]^r - \left[P\left(W_{I_1}(n) = 1\right)\right]^r\right\} \left[P\left(W_{I_1}(n) = 0\right)\right]^{\ell_n - r}$$

$$\le \sum_{r=0}^{\ell_n} \binom{\ell_n}{r} \left[P\left(W_{I_1}(n) \ge 1\right)\right]^r \left[P\left(W_{I_1}(n) = 0\right)\right]^{\ell_n - r}$$

$$- \sum_{r=0}^{\ell_n} \binom{\ell_n}{r} \left[P\left(W_{I_1}(n) = 1\right)\right]^r \left[P\left(W_{I_1}(n) = 0\right)\right]^{\ell_n - r}$$

$$= 1 - \left(1 - P\left(W_{I_1}(n) > 1\right)\right)^{\ell_n}$$

$$= 1 - \left(1 - o\left(\frac{1}{\ell_n}\right)\right)^{\ell_n},$$

which tends to zero.

Before developing the further needed notation and proving the theorem, we state several more useful lemmas which enable us to extend our consideration from events of the form appearing in the condition (2.2.5), that is, of the form $\{\xi_i \le u_n,\ i \in \text{some set of integers } I\}$, to other events relevant to the exceedance problem under our present

procedure.

Lemma 2.2.9. Given an integer $r \geq 1$, let $B_1 \subseteq A_1, \ldots, B_r \subseteq A_r$ be events. Then

$$P[(A_1\backslash B_1)\cap\ldots\cap(A_r\backslash B_r)] = \sum_{s=0}^{r} (-1)^s \sum_{\underline{i}_s} P\left\{\left[\bigcap_{i\notin \underline{i}_s} A_i\right] \cap \left[\bigcap_{i'\in \underline{i}_s} B_{i'}\right]\right\},$$

where for fixed s, $0 \leq s \leq r$, the inner sum is taken over all distinct subsets $\underline{i}_s$ of s integers chosen from among $\{1, \ldots, r\}$. (We define $\underline{i}_o = \phi$.)

Proof: It is trivial that for $r = 1$ or 2 we have respectively,

$$P(A_1\backslash B_1) = P(A_1) - P(B_1)$$

and

$$P[(A_1\backslash B_1) \cap (A_2\backslash B_2)] = P(A_1A_2) - P(A_1B_2) - P(A_2B_1) + P(B_1B_2).$$

Then assuming that the result holds for a given r we have, inductively,

$$\begin{aligned}
&P[(A_1\backslash B_1) \cap \ldots \cap (A_r\backslash B_r) \cap (A_{r+1}\backslash B_{r+1})] \\
&\quad = P[(A_1A_{r+1}\backslash B_1A_{r+1}) \cap \ldots \cap (A_rA_{r+1}\backslash B_rA_{r+1})] \\
&\quad - P[(A_1A_{r+1}\backslash B_1A_{r+1}) \cap \ldots \cap (A_rA_{r+1}\backslash B_rB_{r+1})] \\
&\quad = \sum_{s=0}^{r} (-1)^s \sum_{\underline{i}_s} P\left\{\left[\bigcap_{i\notin \underline{i}_s} A_iA_{r+1}\right] \cap \left[\bigcap_{i'\in \underline{i}_s} B_{i'}A_{r+1}\right]\right\} \\
&\quad - \sum_{s=0}^{r} (-1)^s \sum_{\underline{i}_s} P\left\{\left[\bigcap_{i\notin \underline{i}_s} A_iB_{r+1}\right] \cap \left[\bigcap_{i'\in \underline{i}_s} B_{i'}B_{r+1}\right]\right\}
\end{aligned}$$

$$= \sum_{s=0}^{r+1} (-1)^s \sum_{\underline{i}_s} P\left\{\left[\bigcap_{i \notin \underline{i}_s} A_i\right] \cap \left[\bigcap_{i' \in \underline{i}_s} B_{i'}\right]\right\},$$

where in this last line the inner sum is taken over subsets $\underline{i}_s$ of $\{1, \ldots, r+1\}$. This establishes the lemma.

Lemma 2.2.10. Let $A_1, B_1, \ldots, A_r, B_r$ be events as in Lemma 2.2.9. Then

$$P(A_1 \backslash B_1) \ldots P(A_r \backslash B_r) = \sum_{s=0}^{r} (-1)^s \sum_{\underline{i}_s} \left\{\left[\prod_{i \notin \underline{i}_s} P(A_i)\right]\left[\bigcap_{i' \in \underline{i}_s} P(B_{i'})\right]\right\}.$$

Proof: This is easily obtained by elementary calculations, which will be omitted.

Lemma 2.2.11. Fix n (large) and let $\mathcal{D}_1 \subseteq I_1^{(n)}$, $\mathcal{D}_3 \subseteq I_3^{(n)}$, ..., and $\mathcal{D}^{(n)}_{2\ell_n - 1} \subseteq I^{(n)}_{2\ell_n - 1}$, where the odd-indexed intervals $I_1^{(n)}, \ldots, I^{(n)}_{2\ell_n - 1}$ are as defined in Section 2.1. Now define events D_i by

$$D_i = \{\xi_j \le u_n, \text{ for all } j \in \mathcal{D}_i\},$$

for $i = 1, 3, \ldots, 2\ell_n - 1$. Then (2.2.5) implies

$$(2.2.13) \qquad \left| P\left(\bigcap_{\substack{i=1 \\ i \text{ odd}}}^{2\ell_n - 1} D_j\right) - \prod_{\substack{i=1 \\ i \text{ odd}}}^{2\ell_n - 1} P(D_j)\right| < \ell_n \, g_{n,\beta_n} .$$

Proof: For odd integers i_1, i_2, $1 \le i_1 < i_2 \le 2\ell_n - 1$ we have

$$|P(D_{i_1} D_{i_2}) - P(D_{i_1}) P(D_{i_2})| \le g_{n,\beta_n}$$

by (2.2.5). Now let $1 \le i_1 < \dots < i_{r+1} \le 2\ell_n - 1$ be odd integers. If for $r \le 2$ we have

$$(2.2.14) \qquad |P(\bigcap_{s=1}^{r} D_{i_s}) - \prod_{s=1}^{r} P(D_{i_s})| \le (r-1)g_{n,\beta_n} ,$$

then since for any r the event $\bigcap_{s=1}^{r} D_{i_s}$ is also of the form suitable for applying (2.2.5) we have

$$\begin{aligned} |P(\bigcap_{s=1}^{r+1} D_{i_s}) - \prod_{s=1}^{r+1} P(D_{i_s})| &\le |P(\bigcap_{s=1}^{r+1} D_{i_s}) - P(\bigcap_{s=1}^{r} D_{i_s})P(D_{i_{r+1}})| \\ &\quad + P(D_{i_{r+1}})|P(\bigcap_{s=1}^{r} D_{i_s}) - \prod_{s=1}^{r} P(D_{i_s})| \\ &\le g_{n,\beta_n} + (r-1)g_{n,\beta_n} \\ &= r\, g_{n,\beta_n} . \end{aligned}$$

Thus by induction (2.2.14) holds for $2 \le r \le \ell_n$ and in particular for $r = \ell_n$, which gives (2.2.13).

Combining the previous three lemmas and using the elementary facts that $\sum_{s=0}^{r} \binom{r}{s} = 2^r$ and the absolute value of a sum does not exceed the sum of the absolute values, we obtain the following result.

Lemma 2.2.12. Fix n (large) and an odd integer r such that $1 \le r < 2\ell_n - 1$. Suppose that events A_i, B_i , with $B_i \subseteq A_i$, $i = 1,3, \dots ,r$, and C_i , $i = r+2, r+4, \dots ,2\ell_n - 1$, are all of the same form as the events D_i described in Lemma 2.2.11, that is, of the form $\{\xi_j \le u_n$, for j belonging to some $\mathcal{D}_i \subseteq I_i^{(n)}\}$. Then

$$\left|P\left[(A_1 \setminus B_1) \cap \ldots (A_r \setminus B_r) \cap \left(\bigcap_{\substack{i=r+2 \\ i \text{ odd}}}^{2\ell_n - 1} C_i\right)\right]\right.$$

$$\left. - P(A_1 \setminus B_1) \ldots P(A_r \setminus B_r) P(C_{r+2}) \ldots P(C_{2\ell_n - 1})\right|$$

$$\leq 2^r \ell_n g_{n,\beta_n} .$$

We now establish some further notation needed in the proof of the theorem. Fix n (large) and for each odd integer i, $1 \leq i \leq 2\ell_n - 1$, define events as follows:

$$E_j^i = \{\xi_j > u_n, \; \xi_k \leq u_n, \; k \neq j, \; k \in I_i^{(n)}\}, \quad j \in I_i^{(n)} ,$$

$$F^i = \{\xi_j \leq u_n, \text{ for all } j \in I_i^{(n)}\} = \left\{W_{I_i^{(n)}} = 0\right\} ,$$

and

$$F_j^i = \left\{\xi_k \leq u_n, \text{ for all } k \in I_i^{(n)} \setminus \{j\}\right\} .$$

That is, E_j^i is the event that there is an exceedance of u_n at "time" j in the interval $I_i^{(n)}$ and there are no other exceedances in $I_i^{(n)}$, F^i is the event that there are no exceedances in $I_i^{(n)}$, and F_j^i is the event that there are no exceedances in $I_i^{(n)}$ except possibly at time j. (We have deleted dependence upon n.) It is clear that the following relationships hold:

$$E_j^i \cup F^i = F_j^i ,$$

$$F_j^i \setminus F^i = E_j^i ,$$

and

$$F^i \subseteq F_j^i ,$$

for each $j \in I_i^{(n)}$ and each $i = 1, 3, \ldots, 2\ell_n - 1$. With this notation and the preceding lemmas we now prove the theorem.

<u>Proof of Theorem 2.2.5</u>: Fix n large. We note that for each $i = 1, 3, \ldots, 2\ell_n - 1$,

$$\left(W_{I_i^{(n)}} = 1\right) = \bigcup_{j \in I_i^{(n)}} E_j^i = \bigcup_{j \in I_i^{(n)}} (F_j^i \setminus F^i) ,$$

so that the event $\left(W_{I_i^{(n)}} = 1\right)$ is a disjoint union of α_n subevents each of which is a proper difference of two events, both of the form D_i given in Lemma 2.2.11.

Now, for $0 \le r \le \ell_n$ let $\underline{i}_r = \{i_1, \ldots, i_n\}$ be a choice of r odd integers from $\{1, 3, \ldots, 2\ell_n - 1\}$. Then

$$P\left\{\left[\bigcap_{i \in \underline{i}_r} \left(W_{I_i^{(n)}} = 1\right)\right] \cap \left[\bigcap_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} \left(W_{I_i^{(n)}} = 0\right)\right]\right\}$$

$$= P\left\{\left(\bigcup_{j_{i_1} \in I_{i_1}^{(n)}} E_{j_{i_1}}^{i_1}\right) \cap \cdots \cap \left(\bigcup_{j_{i_r} \in I_{i_r}^{(n)}} E_{j_{i_r}}^{i_r}\right) \cap \left(\bigcap_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} F^{i'}\right)\right\}$$

$$= \sum_{j_{i_1} \in I_{i_1}^{(n)}} \cdots \sum_{j_{i_r} \in I_{i_r}^{(n)}} P\left\{\left(F_{j_{i_1}}^{i_1} - F^{i_1}\right) \cap \cdots \cap \left(F_{j_{i_r}}^{i_r} \setminus F^{i_r}\right) \cap \left(\bigcap_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} F^{i'}\right)\right\} .$$

Also, by stationarity it may be easily seen that

$$\left[P\left(W_{I_1^{(n)}} = 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r}$$

$$= \left[\prod_{i \in \underline{i}_r} P\left(W_{I_i^{(n)}} = 1\right)\right]\left[\prod_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} P\left(W_{I_i^{(n)}} = 0\right)\right]$$

$$= \sum_{j_{i_1} \in I_{i_1}^{(n)}} \cdots \sum_{j_{i_r} \in I_{i_r}^{(n)}} \left\{ P\left(F_{j_{i_1}}^{i_1} \setminus F^{i_1}\right) \cdots P\left(F_{j_{i_r}}^{i_r} \setminus F^{i_r}\right) \cdot \prod_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} P\left(F^{i'}\right)\right\}.$$

Noting that we now have events of the form suitable for applying the development in Lemmas 2.2.8-2.2.12 we see that

$$\left| P\left\{\left[\bigcap_{i \in \underline{i}_r} \left(W_{I_i^{(n)}} = 1\right)\right] \cap \left[\bigcap_{\substack{i' \notin \underline{i}_r \\ i' \text{ odd}}} \left(W_{I_i^{(n)}} = 0\right)\right]\right\} - \left[P\left(W_{I_1^{(n)}} = 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r}\right|$$

$$\leq (2\alpha_n)^r \, \ell_n \, g_{n,\beta_n} ,$$

by Lemma 2.2.12. Then, summing over all possible distinct combinations $\underline{i}_r$ and all r, $0 \leq r \leq k_n - s_n$, and recalling the definition of the event A_n, we have

$$\left| P(A_n) - \sum_{f=0}^{k_n - s_n} \binom{\ell_n}{r} \left[P\left(W_{I_1^{(n)}} = 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r}\right.$$

$$\leq \sum_{r=0}^{k_n - s_n} \binom{\ell_n}{r} (2\alpha_n)^r \, \ell_n \, g_{n,\beta_n}$$

$$< 2^{\ell_n} (2\alpha_n)^{k_n} \, \ell_n \, g_{n,\beta_n}$$

$$< (4\alpha_n)^{\ell_n} \, \ell_n \, g_{n,\beta_n}$$

$$= g_{n,\beta_n} \exp\{\log \ell_n + \ell_n(\log 4 + \log \alpha_n)\} \tag{2.2.15}$$

$$< g_{n,\beta_n} \, e^{n^{\kappa_1}}$$

for all $\kappa_1 > 1 - \lambda > \theta$, when n is sufficiently large. Taking $\kappa_1 > \kappa$ we have this last expression tending to zero, by (2.2.6). Combining this with Lemmas 2.2.6 and 2.2.8(a) we see that $P(A_n) \to \Phi(u)$ as $n \to \infty$, that is, (2.2.7) holds.

To show (2.2.8) we first observe that

$$P(B_n) = P\left[\left(k_n - s_n < W_{\bigcup_{\substack{i=1\\ i\text{ odd}}}^{2\ell_n-1} I_i^{(n)}} < k_n\right) \cap \left(\bigcap_{\substack{i=1\\ i\text{ odd}}}^{2\ell_n-1} \left(W_{I_i^{(n)}} \le 1\right)\right)\right]$$

$$= \sum_{r=k_n-s_n+1}^{k_n-1} \sum_{\underline{i}_r} P\left\{\left[\bigcap_{i\in\underline{i}_r} \left(W_{I_i^{(n)}} = 1\right)\right] \cap \left[\bigcap_{\substack{i'\notin\underline{i}_r\\ i'\text{ odd}}} \left(W_{I_{i'}^{(n)}} = 0\right)\right]\right\},$$

and so, repeating the same steps above, we obtain

$$\left|P(B_n) - \sum_{r=k_n-s_n+1}^{k_n-1} \binom{\ell_n}{r}\left[P\left(W_{I_1^{(n)}} = 1\right)\right]^r \left[P\left(W_{I_1^{(n)}} = 0\right)\right]^{\ell_n - r}\right.$$

$$= \left|\sum_{r=k_n-s_n+1}^{k_n-1} \sum_{\underline{i}_r} P\left\{\left[\prod_{i\in\underline{i}_r} \left(W_{I_i^{(n)}} = 1\right)\right] \cap \left[\bigcap_{\substack{i'\notin\underline{i}_r\\ i'\text{ odd}}} \left(W_{I_{i'}^{(n)}} = 0\right)\right]\right\}\right.$$

$$\left. - \sum_{r=k_n-s_n+1}^{k_n-1} \sum_{\underline{i}_r} \left\{\left[\prod_{i\in\underline{i}_r} P\left(W_{I_i^{(n)}} = 1\right)\right] \cdot \left[\prod_{\substack{i'\notin\underline{i}_r\\ i'\text{ odd}}} P\left(W_{I_{i'}^{(n)}} = 0\right)\right]\right\}\right|$$

$$\le \sum_{r=k_n-s_n+1}^{k_n-1} \binom{\ell_n}{r} (2\alpha_n)^r \, \ell_n \, g_{n,\beta_n}$$

$$< g_{n,\beta_n} \, e^{n^{\kappa_1}}$$

for all $\kappa > \kappa_1 > 1 - \lambda$. This along with Lemmas 2.2.7 and 2.2.8(b)

gives (2.2.8), which completes the proof of the theorem.

Remark 1. In the above development we did not use fully the restriction $\alpha_n k_n^2 = o(n)$ made in (2.2.2), but only that $\alpha_n k_n = o(n)$. With just this latter it turns out that we can extend consideration to intermediate rank sequences k_n for which $\theta < \frac{2}{3}$. However, for independent sequences ξ_n it is easily seen from the relation $P(\xi_1 > u_n) \sim \frac{k_n}{n}$ that the stronger restriction $\alpha_n k_n^2 = o(n)$ is necessary for (2.2.4), so that for an extension our conditions would actually require some degree of departure from independence. Moreover, in the important normal case to be dealt with in Section 2.4, it is appropriate to consider sequences which are at least "approximately independent," and we will find it convenient there to use the stronger restriction.

Remark 2. In the particular case where $\theta = 0$ it is possible to weaken somewhat the exponential "mixing" rate required in (2.2.6), provided that (2.2.4) holds for some α_n for which $\frac{n^\lambda}{\alpha_n} \to 0$ for every $\lambda < 1$ (or more simply, if (2.2.4) holds with α_n replaced by n as the upper limit of summation, a condition which would necessarily exclude independent sequences from consideration). To indicate this, for a given stationary sequence α_n and intermediate rank sequence k_n for which $\theta = 0$, choose $0 < \varepsilon_1 < \frac{1}{2}$ and $\varepsilon_2 > 1$ and let

$$s_n \sim k_n^{\frac{1}{2}-\varepsilon_1},$$

$$\alpha_n \sim \frac{n}{k_n^{1+\varepsilon_2}}, \tag{2.2.16}$$

and

$$\beta_n \sim \frac{n}{k_n^{3/2 + 2\varepsilon_1 + \varepsilon_2}}.$$

so that the relationships (2.2.2) are satisfied. Also, suppose that (2.2.4) is satisfied with this choice of α_n. Then, if (2.2.5) holds with

$$g_{n,\beta_n} = O\left(e^{-k_n^{1+\varepsilon_1+\varepsilon_2} \log n}\right)$$

$$= O\left(n^{-k_n^{1+\varepsilon_1+\varepsilon_2}}\right), \tag{2.2.17}$$

then we can see from (2.2.15) that the conclusion of our theorem remains valid.

As an immediate corollary of the theorem we can obtain a result giving the asymptotic distribution of the k_n-th largest order statistic the stationary sequence ξ_n.

Theorem 2.2.13. Suppose the marginal d.f. of ξ_1 *belongs to the* domain of attraction of the limit law G for the intermediate rank sequence k_n; that is, for some constants $a_n > 0$, b_n we have

$$P\left(\frac{\hat{\xi}_{k_n}^{(n)} - b_n}{a_n} \le x\right) \to G(x) = \Phi(u(x))$$

as $n \to \infty$, for all real x, where $\hat{\xi}_n$ is the independent sequence associated with ξ_n. If A_1 is satisfied for $u_n = a_n x + b_n$ with $u = u(x)$ for all real x for which $u(x)$ is finite, or equivalently, for which $0 < G(x) < 1$, then also

$$P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) \to G(x) \tag{2.2.18}$$

for all real x.

Proof: For x such that $0 < G(x) < 1$ we have the fundamental relation

$$P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) = P(W_n < k_n),$$

where $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$ and $u_n = a_n x + b_n$. Since

$$1 - F(u_n) = \frac{k_n}{n} - \frac{u(x)\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right),$$

by (1.3.4), the conclusion (2.2.18) for such x follows from Theorem 2.2.5. Now suppose x is such that $G(x) = 0$. Since G is continuous, given an arbitrary $\varepsilon > 0$, we can find an $x_\varepsilon > x$ such that $0 < G(x_\varepsilon) < \varepsilon$. Thus

$$\begin{aligned}\lim_n P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) &\le \lim_n P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x_\varepsilon\right) \\ &= G(x_\varepsilon) \\ &< \varepsilon,\end{aligned}$$

so that

$$\lim_n P\left(\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right) = 0 = G(x).$$

Similarly we may consider x such that $G(x) = 1$, and the proof is complete.

We conclude this section by restating the assumption A_1 and Theorem 2.2.13 for consideration of sequences of smallest rank terms $\xi_{k_n}^{(n)}$.

A_1': For the given sequence u_n satisfying

$$F(u_n) = \frac{k_n}{n} + \frac{u\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right)$$

for some real number u, there is a system of sequences $\{\alpha_n, \beta_n, s_n; n\geq 1\}$ subject to (2.1.5), (2.2.1), and (2.2.2), and such that the following are satisfied:

$$n \sum_{j=1}^{\alpha_n - 1} P(\xi_1 \leq u_n, \xi_{1+j} \leq u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

and

For each large n, for integers $p,q \leq \alpha_n$ and integers $1 \leq i_1 < \ldots < i_p < j_1 < \ldots < j_q \leq n$ such that $j_1 - i_p \geq \ell$, we have

$$\begin{aligned} |P(\xi_{i_1} > u_n, \ldots, \xi_{i_p} > u_n, \xi_{j_1} > u_n, \ldots, \xi_{j_q} > u_n) \\ - P(\xi_{i_1} > u_n, \ldots, \xi_{i_p} > u_n) P(\xi_{j_1} > u_n, \ldots, \xi_{j_q} > u_n)| \\ \leq g_{n,\ell}, \end{aligned}$$

where

$$g_{n,\beta_n} = o\left(e^{-n^{\kappa}}\right) \quad \text{as} \quad n \to \infty,$$

for some $\kappa > 1 - \lambda > \theta$, where $\alpha_n \sim n^{\lambda}$ and θ is defined by (2.1.3).

Theorem 2.2.14. Suppose the marginal d.f. of the stationary sequence ξ_n belongs to the domain of attraction of the limit law G for the given intermediate rank sequence k_n, for smallest terms; that is, for some $a_n > 0$, b_n we have

$$P\left(\frac{\hat{\xi}^{(n)}_{k_n} - b_n}{a_n} \leq x\right) \to G(x) = \Phi\left(u(x)\right)$$

as $n \to \infty$, for all real x. If A_1' is satisfied for $u_n = a_n x + b_n$

with $u = u(x)$ for all real x for which $u(x)$ is finite, or equivalently, for which $0 < G(x) < 1$, then also

$$P\left\{\frac{\xi_{k_n}^{(n)} - b_n}{a_n} \le x\right\} \to G(x)$$

for all real x.

2.3. A Second Procedure Using Central Limit Theory.

In this section we present an alternative method for establishing (2.1.2) using the framework in Section 2.1, and in which there is no general restriction as to how rapidly the intermediate rank sequence k_n tends to infinity. The assumption we will make is of a similar type, yet still somewhat different from, the assumption A_1 of the last section, with one important difference being that the relationships (2.2.2), which restricted what intermediate rank sequences could be considered, will be replaced. As we have previously mentioned, the essence of the procedure will be to establish the asymptotic normality of a standardized form of the r.v. W_n in such a way as to lead immediately to the desired conclusion (2.1.2).

As in Section 2.1 we continue to suppose that ξ_n. is a given stationary sequence with marginal d.f. F, k_n is a sequence of intermediate ranks, the level u_n is chosen to satisfy (2.1.1) for some real number u, and $W_n = \sum_{i=1}^{n} I_{[\xi_i > u_n]}$. The following is the assumption to be used in this section.

A_2: There is a system $\{\alpha_n, \beta_n;\ n \ge 1\}$ subject to (2.1.5) and such that

$$\alpha_n = o(\sqrt{k_n})$$

(2.3.1) and

$$\beta_n k_n = o(n),$$

and for which the following hold:

(2.3.2) $$\alpha_n^{-1} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\{P(\xi_1 > u_n,\ \xi_{1+j} > u_n) - (1 - F(u_n))^2\} = o\left(\frac{k_n}{n}\right)$$

as $n \to \infty$, and

(2.3.3) $$\frac{n}{\alpha_n} g_n \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$g_n = \sup_{1 \le m < 2\ell_n - 1} \ \sup_{\substack{A \in \mathcal{F}_{n,m} \\ B \in \mathcal{A}_{n,m+2}}} |P(AB) - P(A)P(B)|,$$

$\mathcal{F}_{n,m}$ being the σ-field generated by $W_{I_1^{(n)}}, \ldots, W_{I_m^{(n)}}$, and $\mathcal{A}_{n,m+2}$ the σ-field generated by $W_{I_{m+2}^{(n)}}, \ldots, W_{I_{2\ell_n}^{(n)}}$.

We note that the intervals $I_m^{(n)}$ and $I_{m+2}^{(n)}$ are "separated" by at least β_n. It is then evident that a condition sufficient for (2.3.3), which is stronger in that more events must be considered, is

(2.3.4) $$\frac{n}{\alpha_n} h_n(\beta_n) \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$h_n(\ell) = \sup_{1 \le m < n - \ell} \ \sup_{\substack{A \in \mathcal{F}_{n,m} \\ B \in \mathcal{A}'_{n,m+\ell+1}}} |P(AB) - P(A)P(B)|,$$

$\mathcal{F}'_{n,m}$ being the σ-field generated by $I_{[\xi_1>u_n]}, \ldots, I_{[\xi_m>u_n]}$ and $\mathcal{F}'_{n,m+\ell+1}$ the σ-field generated by $I_{[\xi_{m+\ell+1}>u_n]}, \ldots, I_{[\xi_n>u_n]}$. Now, for fixed n, m let $\underline{i}$ be a subset of the first m integers and $\underline{j}$ a subset of the integers from $m + \beta_n + 1$ to n. Define events

$$A_{\underline{i}} = \left\{\xi_i>u_n,\ i\in\underline{i};\ \xi_{i'}\leq u_n,\ i'\in\{1,\ldots,m\}\setminus\underline{i}\right\}$$

(2.3.5) and

$$B_{\underline{j}} = \left\{\xi_j>u_n,\ j\in\underline{j};\ \xi_{j'}\leq u_n,\ j'\in\{m+\beta_n+1,\ldots,n\}\setminus\underline{j}\right\}.$$

Then we can see that each $A \in \mathcal{F}_{n,m}$ can be written as some disjoint union of at most 2^m events each of the form $A_{\underline{i}}$, and similarly each event $B \in \mathcal{F}'_{n,m+\beta_n+1}$ is some disjoint union of at most $2^{n-m-\beta_n}$ events of the form $B_{\underline{j}}$. Hence a sufficient condition for (2.3.4) is

$$2^n \frac{n}{\alpha_n} h'_n \to 0, \tag{2.3.6}$$

where

$$h'_n = \sup_{1\leq m<n-\beta_n} \sup_{\underline{i},\underline{j}} |P(A_{\underline{i}}B_{\underline{j}}) - P(A_{\underline{i}})P(B_{\underline{j}})|.$$

One can show, as we did for the condition A_1 in the last section, that (2.3.6) is satisfied by strongly mixing sequences in which $g(\ell) \downarrow 0$ at some particular exponential rate, but that (2.3.6) is potentially much weaker. However, the condition (2.3.3) is substantially weaker yet, there being fewer events to be considered, and apparently an exponential rate of convergence in (2.3.3) is not required. Unfortunately, in some instances (2.3.3) may be hard to verify, due to the nature of the events involved, and as an example, we have found this to be the case for stationary normal sequences considered in

the next section, where with the presently known results dealing with normal sequences we will necessarily be content to work with (2.3.6).

In this section on general stationary sequences we will use assumption A_2 as stated, however, and we now suppose, for the given sequence ξ_n, that sequences of integers α_n and β_n have been chosen for which all of the restrictions and conditions are satisfied. Having this, we then define a triangular array of r.v.'s $\{X_{n,i}\}$, n large and $1 \le i \le 2\ell_n$, by

$$X_{n,i} = \frac{W_{I_i^{(n)}} - E\, W_{I_i^{(n)}}}{\sqrt{\ell_n}\,\sigma_n},$$

where

$$\begin{aligned}\sigma_n^2 &= \operatorname{Var} W_{I_1^{(n)}}\\ &= \alpha_n F(u_n)\bigl(1 - F(u_n)\bigr)\\ &\quad + 2\sum_{j=1}^{\alpha_n-1} (\alpha_n - j)\Bigl\{P(\xi_1>u_n,\ \xi_{1+j}>u_n) - \bigl(1 - F(u_n)\bigr)^2\Bigr\}.\end{aligned} \tag{2.3.7}$$

The key to our approach is the following result of Dvoretzky (1972, Theorem 5.1), a central limit theorem for triangular arrays with dependent r.v.'s in each row, and which extends the classical theorem for arrays with independence within rows (see, for example, Loève (1963, p. 295)). We state this result as a lemma and in original form, which is more general than what we require.

<u>Lemma 2.3.1.</u> Let $\{X_{n,i}\}_{n\ge1,\,1\le i\le N_n\to\infty}$ be an array of r.v.'s with $E\,X_{n,i} = 0$ for all n, i, and let

$$g_n(k) = \sup_{1 \le m < N_n - k} \sup_{\substack{A \in \mathcal{F}_{n,m} \\ B \in \mathcal{A}_{n,m+k+1}}} |P(AB) - P(A)P(B)|,$$

where $\mathcal{F}_{n,m}$ is the σ-field generated by $X_{n,1}, \dots, X_{n,m}$ and $\mathcal{A}_{n,m+k+1}$ is the σ-field generated by $X_{n,m+k+1}, \dots, X_{n,N_n}$. Suppose there are integers

$$0 = j_n(0) < j_n(1) < \dots < j_n(r_n) = N_n$$

such that, defining

$$Y_{n,i} = \sum_{i=j_n(r-1)+1}^{j_n(r)} X_{n,i} , \qquad i = 1, \dots, r_n ,$$

we have

$$\lim_n \sum_{i \text{ odd}} E\, Y_{n,i}^2 = 1,$$

$$\lim_n \sum_{i \text{ even}} E\, Y_{n,i}^2 = 0,$$

and

$$\lim_n \sum_{i=1}^{r_n} E\left(Y_{n,i}^2 \, I_{[|Y_{n,i}| > \varepsilon]}\right) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

Then $\lim_n r_n g_n(k_n) = 0$, where $k_n = \min_{1 \le r \le r_n} \left(j_n(r) - j_n(r-1)\right)$, implies that

$$\sum_{i=1}^{N_n} X_{n,i} \Rightarrow N(0,1).$$

($\Rightarrow$ denotes convergence in distribution.)

Now $W_n = \sum_{i=1}^{2\ell_n} W_{I_i(n)}$, and as the following lemma shows, $\sum_{i=1}^{2\ell_n} X_{n,i}$ is the appropriate standardization of W_n, under the assumption A_2. This result will therefore constitute the major portion of showing that

A_2 implies (2.1.2).

Lemma 2.3.2. Under A_2, $\sum_{i=1}^{2\ell_n} X_{n,i}$ is asymptotically normal (0,1).

Proof: Clearly $E\, X_{n,i} = 0$ for all n,i. If we can establish that

$$(2.3.8) \qquad \lim_n \sum_{i \text{ odd}} E\, X_{n,i}^2 = 1,$$

$$(2.3.9) \qquad \lim_n \sum_{i \text{ even}} E\, X_{n,i}^2 = 0,$$

and

$$(2.3.10) \quad \lim_n \sum_{i=1}^{2\ell_n} E\left(X_{n,i}^2\, I_{[|X_{n,i}|>\varepsilon]}\right) = 0 \quad \text{for every } \varepsilon > 0,$$

then along with (2.3.3) we will have fulfilled the conditions of Lemma 2.3.1, and the conclusion will follow. Here we are taking $N_n = r_n = 2\ell_n$ and $j_n(r) = r$, $0 \le r \le 2\ell_n$, for each n, and as well we are tacitly using the trivial facts that $\mathcal{F}_{n,m}$ is also the σ-field generated by $\{X_{n,i},\ 1 \le i \le m\}$ and $\mathcal{G}_{n,m+2}$ the σ-field generated by $\{X_{n,i},\ m+2 \le i \le 2\ell_n\}$.

We proceed to verify that (2.3.8)-(2.3.10) hold. First we have that

$$\sum_{i \text{ odd}} E\, X_{n,i}^2 = \sum_{i \text{ odd}} \frac{\operatorname{Var} W_{I_i^{(n)}}}{\ell_n\, \sigma_n^2} = 1, \quad \text{for all } n,$$

by stationarity, so that (2.3.8) obviously holds.

Next, using stationarity and for convenience writing $p_n = 1 - F(u_n)$, we have, assuming $\ell_n > 1$,

$$\begin{aligned}\operatorname{Var} W_{I_2^{(n)}} &= E\left\{\sum_{j=1}^{\beta_n}\left(I_{[\xi_j>u_n]} - p_n\right)\right\}^2 \\ &= \beta_n p_n(1-p_n) + 2\sum_{j=1}^{\beta_n-1}(\beta_n-j)\{P(\xi_1>u_n,\ \xi_{1+j}>u_n) - p_n^2\}\end{aligned}$$

$$\le \beta_n p_n(1-p_n) + 2 \sum_{j=1}^{\beta_n - 1} j\, P(\xi_1 > u_n,\ \xi_{1+\beta_n - j} > u_n)$$

$$= \beta_n p_n(1-p_n) + 2 \sum_{j=\alpha_n-\beta_n+1}^{\alpha_n - 1} \frac{j-(\alpha_n-\beta_n)}{j}\, j\, P(\xi_1 > u_n,\ \xi_{\alpha_n - j+1} > u_n)$$

$$\le \beta_n p_n(1-p_n) + \frac{2\beta_n}{\alpha_n - \beta_n} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j) P(\xi_1 > u_n,\ \xi_{1+j} > u_n)$$

$$= \beta_n p_n(1-p_n) + \frac{2\beta_n}{\alpha_n} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\{P(\xi_1 > u_n,\ \xi_{1+j} > u_n) - p_n^2\}$$

$$+ \frac{\beta_n}{\alpha_n - \beta_n} \alpha_n(\alpha_n - 1) p_n^2$$

$$+ 2\left(\frac{\beta_n}{\alpha_n - \beta_n} - \frac{\beta_n}{\alpha_n}\right) \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\{P(\xi_1 > u_n,\ \xi_{1+j} > u_n) - p_n^2\}$$

$$\le \frac{\beta_n}{\alpha_n}\sigma_n^2 + \frac{\beta_n}{1 - \frac{\beta_n}{\alpha_n}} \alpha_n p_n^2 + \frac{2\beta_n^2}{\alpha_n(\alpha_n - \beta_n)} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\{P(\xi_1 > u_n,\ \xi_{1+j} > u_n) - p_n^2\}.$$

From (2.3.2) and the relation $p_n \sim \frac{k_n}{n}$ we have

$$\sigma_n^2 = \alpha_n p_n(1-p_n) + o\left(\frac{\alpha_n k_n}{n}\right) \qquad (2.3.11)$$

$$\sim \frac{\alpha_n k_n}{n},$$

so that

$$\frac{\beta_n \alpha_n p_n^2}{\left(1 - \frac{\beta_n}{\alpha_n}\right)\sigma_n^2} \sim \frac{\beta_n k_n}{n},$$

which tends to zero by (2.3.1), and also

$$\frac{2\beta_n^2}{\sigma_n^2 \alpha_n(\alpha_n - \beta_n)} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\{P(\xi_1 > u_n,\ \xi_{1+j} > u_n) - p_n^2\}$$

$$= o\left(\frac{\beta_n^2}{\alpha_n^2}\right) ,$$

from which we obtain

$$(2.3.12) \qquad \sum_{\substack{i=2 \\ i \text{ even}}}^{2\ell_n - 2} E\, X_{n,i}^2 = \frac{(\ell_n - 1)\operatorname{Var} W_{I_2^{(n)}}}{\ell_n \sigma_n^2} < \frac{\operatorname{Var} W_{I_2^{(n)}}}{\sigma_n^2} \to 0.$$

Also, clearly $E\, X_{n,2\ell_n}^2 = O\left(\frac{\alpha_n + 2\beta_n}{k_n}\right)$, which tends to zero since $\alpha_n = o(k_n)$. This combined with (2.3.12) gives (2.3.9).

Finally, for each $1 \le i \le 2\ell_n$ the r.v. $W_{I_i^{(n)}}$ is bounded above by $\alpha_n + 2\beta_n$, so that $|X_{n,i}|$ is bounded by

$$\frac{\alpha_n + 2\beta_n}{\sqrt{\ell_n}\, \sigma_n} \approx O\left(\frac{\alpha_n}{\sqrt{k_n}}\right) ,$$

which tends to zero. Thus for each $\varepsilon > 0$ we have that

$$\sum_{i=1}^{2\ell_n} E\left(X_{n,i}^2\, I_{[|X_{n,i}| > \varepsilon]}\right) = 0$$

for all sufficiently large n, depending on ε, so that (2.3.10) holds, and hence the proof of the lemma is complete.

The main result is now immediate.

Theorem 2.3.3. Let ξ_n be a stationary sequence of r.v.'s with marginal d.f. F and k_n an intermediate rank sequence. Suppose the level u_n satisfies (2.1.1) for some real u and let W_n be the number of exceedances of u_n by $\xi_1, \dots, \xi_n$. If A_2 holds, then $P(W_n < k_n) \to \Phi(u)$ as $n \to \infty$.

Proof: We have

$$P(W_n < k_n) = P\left(\sum_{i=1}^{2\ell_n} X_{n,i} < \frac{k_n - np_n}{\sqrt{\ell_n}\,\sigma_n} \right) ,$$

where $p_n = 1 - F(u_n)$ and σ_n^2 is given by (2.3.7). Using (2.1.1), (2.1.7), and (2.3.11) we have

$$\frac{k_n - np_n}{\sqrt{\ell_n}\,\sigma_n} \to u \quad \text{as} \quad n \to \infty,$$

and the result follows by Lemma 2.3.2.

We may also state the corresponding theorem giving sufficient conditions under which ${}_{k_n}\xi^{(n)}$ has an asymptotic distribution.

Theorem 2.3.4. Theorem 2.2.13 remains valid if in place of A_1, assumption A_2 holds for $u_n = a_n x + b_n$ with $u = u(x)$ for all x such that $0 < G(x) < 1$.

Finally, we restate our conditions and result for consideration of smallest rank terms $\xi_{k_n}^{(n)}$.

A_2': For the given sequence u_n satisfying

$$F(u_n) = \frac{k_n}{n} + u\,\frac{\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right)$$

for some real number u, there are sequences $\{\alpha_n, \beta_n;\ n \geq 1\}$ subject to (2.1.5) such that $\alpha_n = o(\sqrt{k_n})$ and $\beta_n k_n = o(n)$, for which

$$\alpha_n^{-1} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\{P(\xi_1 \leq u_n,\ \xi_{1+j} \leq u_n) - F^2(u_n)\} = o\left(\frac{k_n}{n}\right)$$

as $n \to \infty$, and (2.3.3) holds, as defined in the assumption A_2.

We then have

Theorem 2.3.5. Theorem 2.2.14 remains valid if in place of A_1', assumption A_2' holds for $u_n = a_n x + b_n$ with $u = u(x)$ for each x such that $0 < G(x) < 1$.

2.4. The Normal Case.

As we noted in Section 1.3 the standard normal law $\Phi(x)$ belongs to its own domain of attraction for any intermediate sequence of right, or left, ranks, and therefore the theorems we have presented in this chapter are applicable to stationary normal sequences. Since for a given value of the mean of ξ_1 a stationary normal sequence ξ_n is specified by its set of covariances, one would hope, just as in the fixed ranks case, to find simple covariance conditions implying our assumptions for general stationary sequences. As might be expected, in order to satisfy the general conditions, which involve the considera-tion of more events than just those relevant to extreme order statis-tics, we will require rates in which the covariances tend to zero that are substantially faster than (1.2.17), (1.2.18), or (1.2.19). For convenience we will deal exclusively with standard (that is, zero mean, unit variance) normal sequences, since results obtained for this case can be easily transformed to other normal sequences.

Thus suppose that ξ_n is a standard stationary normal sequence, that is, ξ_1 has marginal d.f. $\Phi(x)$, and let $r_n = E\,\xi_1\xi_{n+1}$, $n \geq 1$, denote the corresponding sequence of covariances. Let k_n be an intermediate rank sequence and for a given real number x, suppose the level u_n satisfies (2.1.1) with $u = u(x) = x$, that is,

$$1 - \Phi(u_n) = \frac{k_n}{n} - \frac{x\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right) \tag{2.4.1}$$

as $n \to \infty$. One such u_n is of course $u_n = a_n x + b_n$ where a_n and b_n are given by (1.3.19), with $F = \Phi$. Continuing to let W_n denote the number of exceedances of u_n by $\xi_1, \ldots, \xi_n$, we have as our objective to obtain conditions on the covariances r_n under which A_1 or A_2 hold. It will then follow by our previous theorems that $P(W_n < k_n) \to \Phi(x)$, and hence also that the order statistic ${}_{k_n}\xi^{(n)}$ is asymptotically normal.

We will be assuming that the covariances r_n tend to zero as $n \to \infty$, a condition apparently essential in order to have the required "approximate independence" between r.v.'s far apart in time, and it will be convenient to define the quantities

$$\delta = \sup_{n \geq 1} |r_n| \tag{2.4.2}$$

and

$$\delta_n = \sup_{m \geq n} |r_m| . \tag{2.4.3}$$

As has been indicated by Berman (1964), the assumption $r_n \to 0$ implies that $\delta < 1$; this may be seen from the fact that if $\delta = 1$, then $|r_n| = 1$ for some $n \geq 1$, and hence $|r_n| = 1$ infinitely often. Also, it is clear that for each $n \geq 1$ there is an integer $m_n \geq n$ such that $\delta_n = r_{m_n}$; thus if t_n is a sequence of positive numbers monotonically increasing to infinity, and if $r_n t_n$ remains bounded as $n \to \infty$, then since $\delta_n t_n \leq r_{m_n} t_{m_n}$ for each $n \geq 1$, the sequence $\delta_n t_n$ is also bounded.

We begin the development by giving some useful preliminary technical results pertaining to normal sequences. First, using the

familiar relation

$$1 - \Phi(y) \sim \frac{\Phi'(y)}{y} \quad \text{as} \quad y \to \infty,$$

we have, for u_n defined by (2.4.1),

$$\frac{k_n}{n} \sim 1 - \Phi(u_n) \sim \frac{1}{\sqrt{2\pi}\, u_n} e^{-u_n^2/2},$$

so that

$$e^{-u_n^2/2} \sim \sqrt{2\pi}\, u_n \frac{k_n}{n}. \tag{2.4.4}$$

By taking logarithms it follows that

$$u_n^2 - 2 \log \frac{n}{k_n} + 2 \log u_n + \log 2\pi \to 0,$$

or

$$u_n^2 = 2 \log \frac{n}{k_n} + o(u_n^2),$$

so that

$$u_n \sim \sqrt{2 \log \frac{n}{k_n}}. \tag{2.4.5}$$

Hence also, from (2.4.4) and (2.4.5) we have

$$e^{-u_n^2} \sim 4\pi \left(\frac{k_n}{n}\right)^2 \log \frac{n}{k_n}. \tag{2.4.6}$$

We will also require the following result, various forms of which have been used extensively in consideration of extreme order statistics for normal sequences by a number of authors, including Berman (1964) and Leadbetter (1974b, 1977). The precise statement of this result, which we give in a form suitable for our purposes, is implicit in the proof of Lemma 6.6 of Leadbetter (1974a); however for clarity we provide here a sketch of most of the details of proof.

Lemma 2.4.1. Let ξ_n be a standard stationary normal sequence and suppose that $r_n = E\,\xi_1\xi_{n+1} \to 0$ as $n \to \infty$. For $p,q \ge 1$ let $i_1, \ldots, i_p$, $j_1, \ldots, j_q$ be distinct integers. Then for each real number u,

$$|P(\xi_{i_1} \le u, \ldots, \xi_{i_p} \le u, \xi_{j_1} \le u, \ldots, \xi_{j_q} \le u)$$
$$- P(\xi_{i_1} \le u, \ldots, \xi_{i_p} \le u)P(\xi_{j_1} \le u, \ldots, \xi_{j_q} \le u)|$$

$$\le K \sum_{i,j} |\rho_{i,j}|\, e^{-u^2/(1+|\rho_{i,j}|)}, \tag{2.4.7}$$

for some finite constant K depending only on δ, defined by (2.4.2), where $\rho_{i,j} = E\,\xi_i\xi_j = r_{|j-i|}$ for $i \in I = \{i_1,\ldots,i_p\}$, $j \in J = \{j_1,\ldots,j_q\}$, and the summation is taken over all possible pairs $i \in I$, $j \in J$.

Proof: We write

$$P(\xi_{i_1} \le u, \ldots, \xi_{i_p} \le u, \xi_{j_1} \le u, \ldots, \xi_{j_q} \le u)$$
$$= \int_{-\infty}^{u} \cdots \int_{-\infty}^{u} f_1(y_1, \ldots, y_{p+q})\, d(y_1, \ldots, y_{p+q})$$
$$= \int \overset{u}{\underset{-\infty}{\cdots}} \int f_1(\underline{y})\, d\underline{y},$$

where f_1 is the joint density of $\xi_{i_1}, \ldots, \xi_{i_p}$, $\xi_{j_1}, \ldots, \xi_{j_q}$. Now f_1 is based upon the covariance matrix Λ_1, say, which may be partitioned into a 2×2 block matrix whose principal diagonal blocks contain the covariances between the $\xi_{i_1}, \ldots, \xi_{i_p}$ and between the $\xi_{j_1}, \ldots, \xi_{j_q}$, and whose other blocks contain the covariances between ξ_i and ξ_j for all pairs i,j. Now let Λ_0 be the matrix of the

same dimension as Λ_1 and having the same elements in the respective principal diagonal blocks and zeroes elsewhere. For $0 \le h \le 1$ define the matrix $\Lambda_h = h\Lambda_1 + (1-h)\Lambda_0$, and let f_h be the (zero-mean) normal density based on Λ_h. (Λ_h is easily seen to be a covariance matrix.) Let

$$F(h) = \int_{-\infty}^{u} \cdots \int f_h(\underline{y})d\underline{y} ,$$

so that

$$P(\xi_{i_1} \le u, \ldots, \xi_{i_p} \le u, \xi_{j_1} \le u, \ldots, \xi_{j_q} \le u) = F(1)$$

and

$$P(\xi_{i_1} \le u, \ldots, \xi_{i_p} \le u)P(\xi_{j_1} \le u, \ldots, \xi_{j_q} \le u) = F(0).$$

Then the left side of (2.4.7) is $|F(1) - F(0)|$, and we have

$$|F(1) - F(0)| \le \int_0^1 |F'(h)|dh, \tag{2.4.8}$$

where

$$F'(h) = \int_{-\infty}^{u} \cdots \int \frac{\partial f_h(\underline{y})}{\partial h} d\underline{y} .$$

Now if we were to write down explicitly the multidimensional normal density $f_h(\underline{y})$, we would see that it depends on h through the components $\lambda_{i,i'}$ of Λ_h, so that

$$\frac{\partial f_h(\underline{y})}{\partial h} = \sum_{i,i' \in I \cup J} \frac{\partial f_h(\underline{y})}{\partial \lambda_{i,i'}} \cdot \frac{\partial \lambda_{i,i'}}{\partial h} ,$$

and therefore

$$F'(h) = 2 \sum_{i \in I, j \in J} \rho_{i,j} \int_{-\infty}^{u} \cdots \int \frac{\partial f_h(\underline{y})}{\partial \lambda_{i,j}} d\underline{y} ,$$

since $\lambda_{i,i'} = \rho_{i,i'}$ if i and i' are both in I or both in J, and $\lambda_{i,i'} = h\rho_{i,i'}$ if $i \in I$ and $i' \in J$ or $i \in J$ and $i' \in I$. Then, using the fact that the derivative of a multidimensional normal density with respect to a covariance element is equal to the second mixed derivative with respect to the corresponding variables (see Cramer and Leadbetter (1967, Equation 2.10.3)), we obtain

$$|F'(h)| \le \sum_{i\in I, j\in J} |\rho_{i,j}| \left| \int_{-\infty}^{u} \cdots \int \frac{\partial^2 f_h(\underline{y})}{\partial y_i \partial y_j} d\underline{y} \right| ,$$

and integrating y_i and y_j leads to

$$|F'(h)| \le \sum_{i\in I, j\in J} |\rho_{i,j}| \int_{-\infty}^{u} \cdots \int f_h(y_i = y_j = u) d\underline{y}' ,$$

where $f_h(y_i = y_j = u)$ is the function of $p + q - 2$ variables obtained by putting $y_i = y_j = u$ into f_h, and the integration is over *the remaining variables.* For each pair i,j we can dominate this last integration by

$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = y_j = u) d\underline{y}' ,$$

which is just the bivariate normal density, evaluated at (u,u), of two standard normal r.v.'s with covariance $h\rho_{i,j}$. Hence a little obvious estimation gives

$$(2.4.9) \quad |F'(h)| \le \frac{1}{2\pi} \sum_{i\in I, j\in J} |\rho_{i,j}| (1 - \rho_{i,j}^2)^{-\frac{1}{2}} e^{-u^2/(1+|\rho_{i,j}|)}$$

for $0 \le h \le 1$. Now $r_n \to 0$ implies that r_n is bounded away from ± 1, that is, $\delta < 1$, by our earlier remarks, so that $(1 - \rho_{i,j}^2)^{-\frac{1}{2}}$ is bounded above. The conclusion now follows from (2.4.8) and (2.4.9).

By merely making appropriate changes in the limits of integration appearing in the previous proof we can verify the following result.

Lemma 2.4.2. Let ξ_n be a standard stationary normal sequence with covariances $r_n \to 0$. For p_1, p_2, q_1, and $q_2 \geq 1$ let

$$i_1, \dots, i_{p_1}, i'_1, \dots, i'_{p_2}, j_1, \dots, j_{q_1}, j'_1, \dots, j'_{q_2}$$

be distinct integers. Then for each real u,

$$\begin{aligned} |P(\xi_i \leq u,\ i=i_1,\dots,i_{p_1};\ \xi_{i'} > u,\ i'=i'_1,\dots,i'_{p_2};\\ \xi_j \leq u,\ j=j_1,\dots,j_{q_1};\ \xi_{j'} > u,\ j'=j'_1,\dots,j'_{q_2})\\ - P(\xi_i \leq u,\ i=i_1,\dots,i_{p_1};\ \xi_{i'} > u,\ i'=i'_1,\dots,i'_{p_2})\\ \cdot P(\xi_j \leq u,\ j=j_1,\dots,j_{q_1};\ \xi_{j'} > u,\ j'=j'_1,\dots,j'_{q_2})|\\ \leq K \sum_{i,j} |\rho_{i,j}|\, e^{-u^2/(1+|\rho_{i,j}|)} \end{aligned}$$

for some K depending only on δ, where $\rho_{i,j} = E\,\xi_i\xi_j = r_{|j-i|}$ for $i \in I = \{i_1,\dots,i_{p_1}, i'_1,\dots,i'_{p_2}\}$ and $j \in J = \{j_1,\dots,j_{q_1}, j'_1,\dots,j'_{q_2}\}$, and the sum is taken over all pairs of such i,j.

As special cases of the above results we have that for $i < j$ and any real u,

$$(2.4.10)\quad |P(\xi_i \leq u,\ \xi_j \leq u) - \Phi^2(u)| \leq K\,|r_{j-i}|\, e^{-u^2/(1+|r_{j-i}|)}$$

and

$$(2.4.11)\quad |P(\xi_i > u,\ \xi_j > u) - (1-\Phi(u))^2| \leq K\,|r_{j-i}|\, e^{-u^2/(1+|r_{j-i}|)}$$

for some K depending only on δ.

We are now ready to consider the main problem of obtaining covariance conditions for normal sequences under which our general dependence conditions in this chapter are satisfied. In doing this it is most convenient to deal with the assumptions A_1 and A_2 separately, although for each we employ the results just presented in somewhat similar ways. First we consider covariance conditions sufficient for A_1.

For the given standard stationary normal sequence ξ_n with covariances r_n, let us suppose that

$$r_n = O(e^{-n^\rho}) \tag{2.4.12}$$

for some $\rho > 0$. Now assume that for the intermediate rank sequence k_n, with θ defined by (2.1.3), we have satisfied the relations

$$\theta < \frac{1-\delta}{2}, \tag{2.4.13}$$

where $\delta = \sup_{n\geq 1} |r_n|$, and also

$$\theta \leq \frac{\frac{2}{5}\rho}{\rho+1}. \tag{2.4.14}$$

The covariance condition (2.4.12) is of course substantially stronger than those conditions used for the fixed ranks problem, but it is nevertheless reasonable in view of the fairly strong "mixing" condition (2.2.6). For the given normal sequence ξ_n the relations (2.4.13) and (2.4.14) of course impose additional restrictions upon what intermediate rank sequences k_n may be considered, beyond the requirement that $\theta < \frac{2}{5}$, if the ξ_n have high degrees of correlation, that is, δ is "close" to one, or if the specific exponential rate $\rho > 0$ indicating how rapidly r_n tends to zero is small. It thus seems appropriate

that we provide an intuitive explanation as to why these restrictions might be needed. First, we note that (2.2.4) implies that the probability of occurrence of more than one exceedance of u_n in any one of the odd-indexed intervals $I_1, I_3, \ldots, I_{2\ell_n-1}$ constructed in Section 2.1 tends to zero as $n \to \infty$ (Lemma 2.2.3). This suggests that we cannot have simultaneously the level u_n increasing too slowly as $n \to \infty$ and a high degree of correlation between any two r.v.'s ξ_i and ξ_j; for if this were the case we would expect to have at least one and actually more than one exceedance in perhaps several of the odd-indexed intervals. Now it is easily seen that the rate at which u_n increases is inversely related to that of k_n. Hence, for a given normal sequence with high degrees of correlation, we should not expect our construction to be applicable to consideration of intermediate rank sequences k_n for which θ is "large." Secondly, when θ is large our construction requires that λ and μ be chosen small ($\alpha_n \sim n^\lambda$, $\beta_n \sim n^\mu$), according to (2.2.3), and from (2.2.5)-(2.2.6) this makes necessary a more rapid exponential rate of distributional mixing. Then, in making use of Lemma 2.4.1 this corresponds naturally to a faster exponential convergence of the covariances, which, as it turns out for computational purposes, leads to the restriction (2.4.14).

Now, as we have already noted, given the intermediate rank sequence k_n such that $\theta < \frac{2}{5}$, there always can be found a system $\{\alpha_n, \beta_n, s_n; n \geq 1\}$ satisfying (2.1.5), (2.2.1), and (2.2.2), and in fact there is such a system in which $\alpha_n \sim n^\lambda$ for some λ, $0 < \lambda < 1 - 2\theta$, and $\beta_n \sim n^\mu$, $0 < \mu < \lambda$. That (2.2.4) holds with $\alpha_n \sim n^\lambda$ for any λ satisfying $0 < \lambda < 1 - 2\theta$, is easily obtained from the following lemma.

Lemma 2.4.3. If the covariances satisfy (2.4.12) for some $\rho > 0$, and if $\theta < \frac{1-\delta}{2}$, then

$$(2.4.15)\quad n \sum_{j=1}^{n} |r_j| \, e^{-u_n^2/(1+|r_j|)} \to 0 \quad \text{as} \quad n \to \infty.$$

Proof: Choose γ such that $0 < \gamma < \frac{1-2\theta-\delta}{1+\delta}$. We split the summation (2.4.15) into two parts, the first from $j = 1$ to $[n^\gamma]$ and the second from $[n^\gamma] + 1$ to n. ($[\cdot]$ denotes the greatest integer function.) Then by (2.4.6) we have

$$n \sum_{j=1}^{[n^\gamma]} |r_j| \, e^{-u_n^2/(1+|r_j|)} < n^{1+\gamma} e^{-u_n^2/(1+\delta)}$$

$$\le K_1 \, n^{1+\gamma} \left(\log \frac{n}{k_n}\right)^{\frac{1}{1+\delta}} \left(\frac{k_n}{n}\right)^{\frac{2}{1+\delta}}$$

$$< K_2 \, (\log n) \, n^{\frac{\gamma(1+\delta)-1+2\theta+\delta}{1+\delta}} \left(\frac{k_n}{n^\theta}\right)^{\frac{2}{1+\delta}}$$

for some constants K_1 and K_2, and this last quantity tends to zero, by the choice of γ, and since $k_n = o(n^{\theta+\varepsilon})$ for every $\varepsilon > 0$. Also we have

$$n \sum_{j=[n^\gamma]+1}^{n} |r_j| \, e^{-u_n^2/(1+|r_j|)} \le n^2 \, \delta_{[n^\gamma]} \, e^{-u_n^2/(1+\delta_{[n^\gamma]})}$$

$$\le K_1 \, n^2 \delta_{[n^\gamma]} \left(\log \frac{n}{k_n}\right)^{1/(1+\delta_{[n^\gamma]})} \left(\frac{k_n}{n}\right)^{2/(1+\delta_{[n^\gamma]})}$$

$$< K_2 \, n \, k_n (\log n) \, \delta_{[n^\gamma]} ,$$

and since $\delta_{[n^\gamma]} = O\left(e^{-n^{\gamma\rho}}\right)$, the last expression tends to zero. Combining both sums gives the conclusion.

Lemma 2.4.4. Under the conditions of the previous lemma,

$$n \sum_{j=1}^{\alpha_n-1} P(\xi_1>u_n,\ \xi_{1+j}>u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

where $\alpha_n \sim n^\lambda$, for any choice of λ such that $0 < \lambda < 1 - 2\theta$.

Proof: Using (2.4.11) we have for some finite K,

$$n \sum_{j=1}^{\alpha_n-1} P(\xi_1>u_n,\ \xi_{1+j}>u_n) \le n\,\alpha_n\bigl(1-\Phi(u_n)\bigr)^2 + K\,n \sum_{j=1}^{\alpha_n-1} |r_j|\, e^{-u_n^2/(1+|r_j|)}$$

$$\le n\,\alpha_n\bigl(1-\Phi(u_n)\bigr)^2 + K\,n \sum_{j=1}^{n} |r_j|\, e^{-u_n^2/(1+|r_j|)},$$

for all large n. The first term on the right tends to zero, since $1-\Phi(u_n) \sim \frac{k_n}{n}$ and $\alpha_n k_n^2 = o(n)$, and the result follows from the previous lemma.

The next result provides a sufficient condition for (2.2.5)-(2.2.6) to hold, for given sequences of integers α_n and β_n.

Lemma 2.4.5. (2.2.5)-(2.2.6) is satisfied if β_n is such that $\delta_{\beta_n} = O\left(e^{-n^{\kappa_0}}\right)$ for some $\kappa_0 > 1 - \lambda$, where $\alpha_n \sim n^\lambda$, $0 < \lambda < 1$.

Proof: For integers $1 \le i_1 < \dots < i_p < j_1 < \dots < j_q \le n$ we have by Lemma 2.4.1,

$$\bigl|P\bigl(\xi_{i_1}\le u_n, \dots, \xi_{i_p}\le u_n, \xi_{j_1}\le u_n, \dots, \xi_{j_q}\le u_n\bigr)$$

$$- P\bigl(\xi_{i_1}\le u_n, \dots, \xi_{i_p}\le u_n\bigr)P\bigl(\xi_{j_1}\le u_n, \dots, \xi_{j_q}\le u_n\bigr)\bigr|$$

$$\le K \sum_{i,j} |\rho_{i,j}|\, e^{-u_n^2/(1+|\rho_{i,j}|)},$$

for some constant K, where $\rho_{i,j} = E\,\xi_i\xi_j$ for $i = i_1, \dots, i_p$ and

$j = j_1, \ldots, j_q$. But for $j_1 - i_p \geq \beta_n$ we see that this quantity does not exceed

$$K\, n^2\, \delta_{\beta_n}\, e^{-u_n^2/(1+\delta_{\beta_n})} \leq K_1\, n^2\, \delta_{\beta_n} \left(\log \frac{n}{k_n}\right)^{1/(1+\delta_{\beta_n})} \left(\frac{k_n}{n}\right)^{2/(1+\delta_{\beta_n})}$$

$$\leq K_2\, (\log n)\, \delta_{\beta_n}\, nk_n$$

for some constants K_1 and K_2 , and for all large n. Thus if $\delta_{\beta_n} = O\left(e^{-n^{\kappa_0}}\right)$ for some $\kappa_0 > 1 - \lambda$, then the last expression above may be written as $O\left(e^{-n^{\kappa}}\right)$ for some κ, $\kappa_0 > \kappa > 1 - \lambda$.

It only remains for us to indicate, under our conditions and restrictions, how the system of integer sequences $\{\alpha_n, \beta_n, s_n;\, n \geq 1\}$ may be chosen in such a way that the hypothesis of the previous lemma is satisfied. We do this and summarize our discussion of obtaining covariance conditions sufficient for A_1 in the following theorem.

Theorem 2.4.6. Let ξ_n be a standard stationary normal sequence with covariances $r_n = E\, \xi_1 \xi_{n+1}$ such that $r_n = O\left(e^{-n^{\rho}}\right)$ for some $\rho > 0$. Let k_n be an intermediate rank sequence such that $\theta = \theta(\{k_n\}) < \frac{1-\delta}{2}$, where $\delta = \sup_{n \leq 1} |r_n|$, and also $\theta < \frac{2/5\ \rho}{\rho+1}$. Suppose for a given real number x the level u_n satisfies

$$1 - \Phi(u_n) = \frac{k_n}{n} - \frac{x\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right) .$$

Then assumption A_1 holds, so that by Theorem 2.2.5 we have

$$P(W_n < k_n) \to \Phi(x) \quad \text{as} \quad n \to \infty,$$

where W_n is the number of exceedances of u_n by $\xi_1, \ldots, \xi_n$.

Proof ($\theta > 0$): Take $\alpha_n \sim n^{\lambda}$ and $\beta_n \sim n^{\mu}$ where $0 < \mu < \lambda < 1$ are chosen such that $\mu\rho > 1 - \lambda > 0$, and let the sequence $s_n = o(\sqrt{k_n})$ be chosen arbitrarily. Then by Lemma 2.4.4, (2.2.4) holds, and since $\delta_{\beta_n} = O\left(e^{-n^{\mu\rho}}\right)$ where $\mu\rho > 1 - \lambda$, (2.2.5)-(2.2.6) is satisfied, by Lemma 2.4.5.

($\theta > 0$): Let $0 < \varepsilon < \min\{1 - \frac{5}{2}\theta, \frac{5}{4}\theta(\frac{2}{5} - \theta)\}$ and choose $\mu = 1 - \frac{5}{2}\theta - \varepsilon$ and $\lambda = 1 - 2\theta - \frac{\varepsilon}{3}$. Also, let $s_n \sim n^{\nu}$, where $\nu = \frac{\theta}{2} - \frac{\varepsilon}{3} > 0$. Then the relationships (2.2.2) are satisfied for $\alpha_n \sim n^{\lambda}$ and $\beta_n \sim n^{\mu}$, and by Lemma 2.4.4, (2.2.4) holds. Moreover, $\delta_{\beta_n} = O\left(e^{-n^{\mu\rho}}\right)$ where $\rho \geq \dfrac{\theta}{\frac{2}{5} - \theta}$, and hence

$$\begin{aligned}
\mu\rho &\geq (1 - \tfrac{5}{2}\theta - \varepsilon)\,\frac{\theta}{\frac{2}{5} - \theta} \\
&= \frac{5}{2}\theta - \frac{\theta}{\frac{2}{5} - \theta} \\
&> 2\theta + \varepsilon \\
&> 1 - \lambda.
\end{aligned}$$

Thus with $\kappa_0 = \mu\rho$ we see that (2.2.5)-(2.2.6) holds by Lemma 2.4.5. This completes the proof.

It is clear that the system $\{\alpha_n, \beta_n, s_n; n \geq 1\}$ as chosen does not depend on x. The following result giving the asymptotic distribution of the k_n-th largest order statistic $_{k_n}\xi^{(n)}$ is thus an immediate corollary.

Theorem 2.4.7. Under the same covariance condition and restrictions on k_n as in the previous theorem, for each real x we have

$$P\left(\frac{k_n \xi^{(n)} - b_n}{a_n} \le x\right) \to \Phi(x)$$

as $n \to \infty$, where a_n and b_n are given by (1.3.19), with $F = \Phi$, that is,

$$\Phi(b_n) = 1 - \frac{k_n}{n}$$

and

$$a_n = \frac{\sqrt{k_n}}{n\Phi'(b_n)} .$$

Remark: It is clear that for consideration of intermediate rank sequences k_n for which $\theta = 0$, if (2.4.12) holds, then (2.4.13) and (2.4.14) are automatically satisfied. In fact, it is possible in this case to weaken somewhat the requirement of an exponential rate, in accordance with Remark 2 in Section 2.2. Suppose for the system $\{\alpha_n, \beta_n, s_n;\ n\ge 1\}$ given by (2.2.16) we have that

$$r_n = O\left(\frac{(\beta^{-1}(n))^{-\left(k_{\beta^{-1}(n)}^{1+\varepsilon_1+\varepsilon_2} + 1\right)} k_{\beta^{-1}(n)}^{1+2\varepsilon_2}}{\log \beta^{-1}(n)}\right)$$

where $\beta^{-1}(n)$ is an integer such that $\beta_{\beta^{-1}(n)} = n$. It follows that

$$\delta_{\beta_n} = O\left(\frac{n^{-\left(k_n^{1+\varepsilon_1+\varepsilon_2} + 1\right)} k_n^{1+2\varepsilon_2}}{\log n}\right) ,$$

and moreover we see readily that

$$\delta_{[n^\gamma]} = O\left(\frac{n^{-\gamma k_{[n^\gamma]}} k_n^{1+2\varepsilon_2}}{\log n}\right)$$

for each $0 < \gamma < 1$. An inspection of the proofs of Lemmas 2.4.3-2.4.5 reveals that we can now verify (2.2.4) and (2.2.5), with the rate (2.2.17).

We now turn to the task of developing covariance conditions under which the assumption A_2 holds. In doing this we will employ methods similar to those we have seen for obtaining conditions sufficient for A_1, using our established technical results applying to normal sequences. The exponential rate in which the covariances tend to zero will again be appropriate, this in the first place resulting from the "slow" rate in which the level u_n increases, and secondly owing to the fact that we will deal with the condition (2.3.6), rather than (2.3.3) directly. We will also impose certain restrictions upon what intermediate rank sequences k_n can be considered, for a specified exponential rate, but these will be substantially different from (2.4.13) and (2.4.14), in that we admit sequences k_n for which θ is "large."

Thus again let ξ_n be a standard stationary normal sequence. We suppose now that the covariances r_n satisfy

(2.4.16) $$r_n = O(e^{-n^\rho}) \quad \text{for some } \rho > 3.$$

Let k_n be an intermediate rank sequence, subject initially to only the restriction that $\theta < 1$, and for a given real x choose the level u_n to satisfy (2.4.1). As the following result shows, the condition (2.3.2) holds for any sequences α_n, β_n satisfying (2.3.1).

Lemma 2.4.8. If the covariances satisfy (2.4.16) and if for the intermediate rank sequence k_n such that $\theta < 1$, the sequences α_n and β_n

are chosen subject to (2.3.1), then (2.3.2) holds for u_n given by (2.4.1), for any real x; in fact, for each such u_n we have

$$\frac{n}{k_n} \sum_{j=1}^{[\sqrt{k_n}]} |P(\xi_1 > u_n, \xi_{1+j} > u_n) - (1 - \Phi(u_n))^2| \to 0, \quad \text{as} \quad n \to \infty.$$

Proof: Since $r_n \to 0$ we have $\delta < 1$, and since $\theta < 1$, the quantity $1 - \theta - \delta + \theta\delta = (1-\delta)(1-\theta)$ is positive. Thus we may choose γ such that $0 < \gamma < \frac{1-\theta-\delta+\theta\delta}{1+\delta}$. Splitting the given summation into two parts, as in the proof of Lemma 2.4.3, we have first that

$$\frac{n}{k_n} \sum_{j=1}^{[n^\gamma]} |P(\xi_1 > u_n, \xi_{1+j} > u_n) - (1 - \Phi(u_n))^2|$$

$$\leq K \frac{n}{k_n} \sum_{j=1}^{[n^\gamma]} e^{-u_n^2/(1+|r_j|)}$$

for some constant K, by (2.4.11), and by (2.4.6) the quantity on the right side does not exceed

$$K \frac{n^{1+\delta}}{k_n} e^{-u_n^2/(1+\delta)} \leq K_1 \frac{n^{1+\gamma}}{k_n} \left(\log \frac{n}{k_n}\right)^{1/(1+\delta)} \left(\frac{k_n}{n}\right)^{2/(1+\delta)}$$

$$< K_2 (\log n)\, n^{\theta\left(\frac{2}{1+\delta} - 1\right)+1+\gamma-2/(1+\delta)} \left(\frac{k_n}{n^\theta}\right)^{2/(1+\delta)-1}$$

$$= K_2 (\log n)\, n^{\frac{\gamma(1+\delta)-1+\delta+\theta-\theta\delta}{1+\delta}} \left(\frac{k_n}{n^\theta}\right)^{\frac{1-\delta}{1+\delta}}$$

for some constants K_1 and K_2. Then by (2.1.4) and the choice of γ, this expression tends to zero. The proof is complete if it is possible to choose γ such that $\sqrt{k_n} \leq n^\gamma$ for all large n. Otherwise for n such that $\sqrt{k_n} > n^\gamma$ we have

$$\frac{n}{k_n} \sum_{j=[n^\gamma]+1}^{[\sqrt{k_n}]} \left|P(\xi_1>u_n,\ \xi_{1+j}>u_n) - \left(1 - \Phi(u_n)\right)^2\right|$$

$$\leq K \frac{n}{\sqrt{k_n}} \delta_{[n^\gamma]}\ e^{-u_n^2/(1+\delta_{[n^\gamma]})}$$

$$< K_1 \sqrt{k_n}\ (\log n)\ \delta_{[n^\gamma]}\ ,$$

which clearly tends to zero, since $\delta_{[n^\gamma]} = O\left(e^{-n^{\gamma\rho}}\right)$. The result then follows upon combining the two summations.

We now show that (2.3.6) holds for the level u_n given by (2.4.1) if the covariances r_n satisfy (2.4.16). To do this we make the additional restriction upon k_n that

$$\frac{2}{\rho} < \theta < \frac{\rho-1}{\rho}\ .$$

(Note $\rho > 3$.) Then we may choose sequences of integers $\alpha_n \sim n^\lambda$ and $\beta_n \sim n^\mu$, where

(2.4.17) $$0 < \frac{1}{\rho} < \mu < \lambda < \frac{\theta}{2}$$

and

$$\mu + \theta < 1.$$

It is clear that the relationships (2.3.1) are satisfied, and by the previous lemma, (2.3.2) holds. Next, we fix n (large) and m, $1 \leq m < n - \beta_n$, and let $A_{\underline{i}_r}$ and $B_{\underline{j}_s}$ be events of the form (2.3.5), that is,

$$A_{\underline{i}_r} = \left\{\xi_i>u_n,\ i\in\underline{i}_r,\ \xi_{i'}\leq u_n,\ i'\in\{1,\ldots,m\}\setminus\underline{i}_r\right\}$$

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and

$$B_{\underline{j}_s} = \left\{\xi_j > u_n,\ j \in \underline{j}_s,\ \xi_{j'} \le u_n,\ j' \in \{m+\beta_n+1,\dots,m\} \setminus \underline{j}_s\right\}$$

where $0 \le r \le m$, $0 \le s \le n - m + \beta_n$, $\underline{i}_r$ is a subset of r of the first m positive integers, and $\underline{j}_s$ is a subset of s of the integers from $m + \beta_n + 1$ to n. Then by Lemma 2.4.2 we have

$$|P(A_{\underline{i}_r} B_{\underline{j}_s}) - P(A_{\underline{i}_r})P(B_{\underline{j}_s})| \le K \sum_{i,j} |\rho_{i,j}|\, e^{-u_n^2/(1+|\rho_{i,j}|)}$$

where $\rho_{i,j} = E\,\xi_i\xi_j$ and the summation is taken over all pairs i,j such that $1 \le i \le m$ and $m + \beta_n + 1 \le j \le n$. The right side above not exceed

$$K\, n^2\, \delta_{\beta_n}\, e^{-u_n^2/(1+\delta_{\beta_n})} < K_1\, (\log n) n\, k_n\, \delta_{\beta_n} .$$

But $\delta_{\beta_n} = O(e^{-n^{\mu\rho}})$, and by (2.4.17), $\mu\rho > 1$. Then taking the supremum over all choices of indices $\underline{i}_r$, $\underline{j}_s$, and all m, $1 \le m < n - \beta_n$, we see that (2.3.6) holds, which implies (2.3.3). Summarizing, we have proven the following result.

Theorem 2.4.9. Suppose ξ_n is a standard stationary normal sequence with covariances $r_n = \xi_1\xi_{n+1}$, $n \ge 1$, satisfying

$$r_n = O(e^{-n^\rho})$$

for some $\rho > 3$. Let k_n be an intermediate rank sequence for which

$$\frac{2}{\rho} < \theta < \frac{\rho-1}{\rho}$$

where θ is defined by (2.1.3), and suppose the level u_n satisfies

$$1 - \Phi(u_n) = \frac{k_n}{n} - \frac{x\sqrt{k_n}}{n} + o\left(\frac{\sqrt{k_n}}{n}\right)$$

for some real x. Then A_2 holds, so that by Theorem 2.3.3,

$$P(W_n < k_n) \to \Phi(x)$$

as $n \to \infty$, where W_n is the number of exceedances of u_n by $\xi_1, \ldots, \xi_n$. Moreover, it follows that

$$P\left(\frac{\xi^{(n)}_{k_n} - b_n}{a_n} \leq x\right) \to \Phi(x)$$

for all real x, where the normalizing constants a_n, b_n are given by

$$\Phi(b_n) = 1 - \frac{k_n}{n}$$

and

$$a_n = \frac{\sqrt{k_n}}{n\Phi'(b_n)} .$$

Finally we mention that results corresponding to the last statement of the previous theorem as well as to Theorem 2.4.7 can be obtained for consideration of smallest rank terms, and we state these formally.

Theorem 2.4.10. For the standard stationary normal sequence ξ_n, if the covariance conditions of either Theorem 2.4.6 or Theorem 2.4.9 are satisfied, along with the respective restrictions on k_n, then

$$P\left(\frac{\xi^{(n)}_{k_n} - b_n}{a_n} \leq x\right) \to \Phi(x) \quad \text{as} \quad n \to \infty,$$

for all real x, where a_n and b_n are given by

$$\Phi(b_n) = \frac{k_n}{n}$$

and

$$a_n = \frac{\sqrt{k_n}}{n\Phi'(b_n)} .$$

CHAPTER III

JOINT DISTRIBUTIONS AND THE MULTI-LEVEL EXCEEDANCE PROBLEM

We now examine the problem of obtaining the asymptotic joint distribution of two or more order statistics from the same ξ_n sequence, as well as the related problem involving the limiting joint distribution of the numbers of exceedances by ξ_n of two or more appropriately determined levels. We first deal with the i.i.d. sequence case, where several results are already known, and then we turn to the case in which the ξ_n are identically distributed but exhibit dependence of the type we have previously considered.

3.1. The I.i.d. Case.

In Section 1.2 we discussed the asymptotic distribution of two or more fixed largest or fixed smallest rank order statistics from i.i.d. sequences. Other results giving the asymptotic distribution of two or more central rank terms, as well as the limiting distribution of several fixed rank terms and central rank terms considered jointly, have also been obtained by Loève (1956). In each of these instances it is assumed that the corresponding marginal or one-dimensional limiting distributions exist. Moreover, as is easily seen from Loève's results, the expressions obtained for the limiting joint distributions of several fixed smallest and largest rank terms and central rank terms may be factored into the corresponding distributions for these three classes

of rank sequences; that is, the largest rank terms, smallest rank terms, and central rank terms are asymptotically mutually independent, when the limiting distributions exist individually.

We have been able to extend these results to include intermediate order statistics, and for smallest and largest rank intermediate terms simultaneously. Our conditions also include the existence of the corresponding one-dimensional limit laws, as well as a restriction upon the several intermediate rank sequences being considered. As is clear from the first and our main new result below, intermediate terms and central terms are asymptotically mutually independent.

<u>Theorem 3.1.1</u>. Let m_1 , m_2 , and m_3 be nonnegative integers and $m = m_1 + m_2 + m_3$. Let $k_n^{(1)}, \ldots, k_n^{(m)}$ be m increasing rank sequences ($k_n \to \infty$, $n - k_n \to \infty$) such that, for some integer $N \geq 1$, $k_n^{(1)} \leq \ldots \leq k_n^{(m)}$ for all $n \geq N$, and suppose

$$\lim_n \frac{k_n^{(\mu)}}{n} = 0, \qquad \mu = 1, \ldots, m_1 ,$$

$$\lim_n \frac{k_n^{(m_1+\mu)}}{n} = \lambda_{m_1+\mu} , \qquad \mu = 1, \ldots, m_2 ,$$

where

$$0 < \lambda_{m_1+1} \leq \ldots \leq \lambda_{m_1+m_2} < 1,$$

and

$$\lim_n \frac{k_n^{(m_1+m_2+\mu)}}{n} = 1, \qquad \mu = 1, \ldots, m_3 .$$

Assume also that

$$\lambda_{\mu,\nu} = \lim_n \frac{k_n^{(\mu)}}{k_n^{(\nu)}}$$

exists for each $1 \le \mu \le \nu \le m_1$ $(0 \le \lambda_{\mu,\nu} \le 1)$, and that

$$\lambda_{m_1+m_2+\mu,m_1+m_2+\nu} = \lim_n \frac{n-k_n^{(m_1+m_2+\nu)}}{n-k_n^{(m_1+m_2+\mu)}}$$

exists for each $1 \le \mu \le \nu \le m_3$ $(0 \le \lambda_{m_1+m_2+\mu,m_1+m_2+\nu} \le 1)$. Now let ξ_n be an i.i.d. sequence with marginal d.f. F and suppose that F belongs to the domain of attraction for k_n-th smallest terms of the law

$$G^{(\mu)}(x) = \Phi\left(u^{(\mu)}(x)\right),$$

for each $\mu = 1, \ldots ,m$. That is, there are sequences $a_n^{(\mu)} > 0$, $b_n^{(\mu)}$ such that

$$P\left(\frac{\xi_{k_n^{(\mu)}}^{(n)} - b_n^{(\mu)}}{a_n^{(\mu)}} \le x\right) \to G^{(\mu)}(x) \tag{3.1.1}$$

as $n \to \infty$, for all continuity points x of $G^{(\mu)}$, $\mu = 1, \ldots ,m$. (Recall that the continuity set of $G^{(\mu)}$ for $\mu = 1, \ldots ,m_1, m_1+m_2+1, \ldots ,m$ is the entire real line.) Then for all real $x_1, \ldots ,x_m$ for which $x_{m_1+\mu}$ is a continuity point of $G^{(m_1+\mu)}$, $\mu = 1, \ldots ,m_2$,

$$P\left(\frac{\xi_{k_n^{(1)}}^{(n)} - b_n^{(1)}}{a_n^{(1)}} \le x_1, \ldots ,\frac{\xi_{k_n^{(m)}}^{(n)} - b_n^{(m)}}{a_n^{(m)}} \le x_m\right)$$

$$\to \Phi_\Sigma \left(u^{(1)}(x_1), \ldots ,u^{(m)}(x_m)\right), \tag{3.1.2}$$

where Φ_Σ is the m-dimensional normal law which has mean the m-dimensional zero vector, and whose covariance is the m×m-dimensional block diagonal matrix Σ given by

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},$$

where Σ_1 is the symmetric $m_1 \times m_1$ matrix whose (μ,ν)-th component is $\lambda^{1/2}_{\mu,\nu}$, $1 \le \mu \le \nu \le m_1$, Σ_2 is the symmetric $m_2 \times m_2$ matrix whose (μ,ν)-th component is

$$\left\{\frac{\lambda_{m_1+\mu}(1-\lambda_{m_1+\nu})}{\lambda_{m_1+\nu}(1-\lambda_{m_1+\mu})}\right\}^{1/2},$$

$1 \le \mu \le \nu \le m_2$, and Σ_3 is the symmetric $m_3 \times m_3$ matrix whose (μ,ν)-th component is $\lambda^{1/2}_{m_1+m_2+\mu,m_1+m_2+\nu}$, $1 \le \mu \le \nu \le m_3$.

It is clear that we may alternatively write the limiting distribution (3.1.2) as

$$\Phi_{\Sigma_1}\left(u^{(1)}(x_1), \ldots, u^{(m_1)}(x_{m_1})\right) \cdot \Phi_{\Sigma_2}\left(u^{(m_1+1)}(x_{m_1+1}), \ldots, u^{(m_1+m_2)}(x_{m_1+m_2})\right) \cdot \Phi_{\Sigma_3}\left(u^{(m_1+m_2+1)}(x_{m_1+m_2+1}), \ldots, u^{(m)}(x_m)\right),$$

where Φ_{Σ_j} is the zero-mean multinormal law with covariance matrix Σ_j, $j = 1,2,3$.

<u>Proof</u>: Let $x_1, \ldots, x_n$ be real numbers for which $u^{(\mu)}(x_\mu)$ is finite for each $\mu = 1, \ldots, m$ and for which $x_{m_1+\mu}$ is a continuity point of $G^{(m_1+\mu)}$, $\mu = 1, \ldots, m_2$. For each $\mu = 1, \ldots, m$ let $a_n^{(\mu)} > 0$, $b_n^{(\mu)}$ be such that (3.1.1) holds for all continuity points x of $G^{(\mu)}$. We may write

$$P\left(\frac{\xi^{(n)}_{k_n^{(1)}} - b_n^{(1)}}{a_n^{(1)}} \le x_1, \ldots, \frac{\xi^{(n)}_{k_n^{(m)}} - b_n^{(m)}}{a_n^{(m)}} \le x_m\right)$$

(3.1.3) $$= P\left(X_n^{(1)}(x_1) \le v_n^{(1)}(x_1), \ldots, X_n^{(m)}(x_m) \le v_n^{(m)}(x_m)\right),$$

where

$$\chi_n^{(\mu)}(x_\mu) = -\sum_{j=1}^{n} \frac{I_{[\xi_j \le a_n^{(\mu)} x_\mu + b_n^{(\mu)}]} - p_n^{(\mu)}(x_\mu)}{\sqrt{np_n^{(\mu)}(x_n) q_n^{(\mu)}(x_\mu)}} ,$$

$$p_n^{(\mu)}(x_\mu) = F\left(a_n^{(\mu)} x_\mu + b_n^{(\mu)}\right),$$

$$q_n^{(\mu)}(x_\mu) = 1 - p_n^{(\mu)}(x_\mu),$$

and

$$v_n^{(\mu)}(x_\mu) = \frac{np_n^{(\mu)}(x_\mu) - k_n^{(\mu)}}{\sqrt{np_n^{(\mu)}(x_\mu) q_n^{(\mu)}(x_\mu)}}$$

$\mu = 1, \ldots, m$. Since we will assume throughout the proof that $x_1, \ldots, x_m$ are fixed real numbers, for notational simplicity we write $\chi_n^{(\mu)}$, $p_n^{(\mu)}$, $q_n^{(\mu)}$, and $v_n^{(\mu)}$ for $\chi_n^{(\mu)}(x_\mu)$, $p_n^{(\mu)}(x_\mu)$, $q_n^{(\mu)}(x_\mu)$, and $v_n^{(\mu)}(x_\mu)$ respectively.

From (1.3.1b) we have the relation

$$p_n^{(\mu)} = \frac{k_n^{(\mu)}}{n} + \frac{u^{(\mu)}(x_\mu)\sqrt{k_n^{(\mu)}\left(1 - \frac{k_n^{(\mu)}}{n}\right)}}{n}\left(1 + o(1)\right),$$

for each $\mu = 1, \ldots, m$, so that

$$np_n^{(\mu)} \sim k_n^{(\mu)} ,$$

and moreover,

$$q_n^{(\mu)} \sim 1 - \frac{k_n^{(\mu)}}{n} ,$$

since

$$\frac{1 - \frac{k_n^{(\mu)}}{n} - \frac{u^{(\mu)}(x_\mu)\sqrt{k_n^{(\mu)}\left[1 - \frac{k_n^{(\mu)}}{n}\right]}}{n}\left(1+o(1)\right)}{1 - \frac{k_n^{(\mu)}}{n}}$$

$$= 1 - u^{(\mu)}(x_\mu)\sqrt{\frac{k_n^{(\mu)}}{n\left(n-k_n^{(\mu)}\right)}}\left(1 + o(1)\right) \to 1$$

as $n \to \infty$. Thus for each $\mu = 1, \ldots, m$ we have that

$$v_n^{(\mu)} \to u^{(\mu)}(x_\mu) \ . \tag{3.1.4}$$

Now according to the well-known Cramér-Wold theorem for convergence in distribution of random vectors, the relation

$$\mathcal{L}((X_n^{(1)}, \ldots, X_n^{(m)}) \to \mathcal{L}((X^{(1)}, \ldots, X^{(m)})$$

holds if and only if

$$\mathcal{L}(c_1 X_n^{(1)} + \ldots + c_m X_n^{(m)}) \to \mathcal{L}(c_1 X^{(1)} + \ldots + c_m X^{(m)})$$

for all real $c_1, \ldots, c_m$.

Thus let $c_1, \ldots, c_m$ be given. We may write

$$c_1 X_n^{(1)} + \ldots + c_m X_n^{(m)} = \sum_{j=1}^{n} \hat{X}_{n,j} \ ,$$

say, where

$$\hat{X}_{n,j} = -\sum_{\mu=1}^{m} c_\mu \frac{I_{[\xi_j \le u_n^{(\mu)}]} - p_n^{(\mu)}}{\sqrt{n p_n^{(\mu)} q_n^{(\mu)}}}$$

$j = 1, \ldots, n$, and where

$$u_n^{(\mu)} = u_n^{(\mu)}(x_\mu) = a_n^{(\mu)} x_\mu + b_n^{(\mu)} \ ,$$

$\mu = 1, \ldots, m$.

Consider the array $\{\hat{X}_{n,j}\}_{n \ge 1, 1 \le j \le n}$. Clearly the r.v.'s $\hat{X}_{n,1}, \ldots, \hat{X}_{n,n}$ are independent, since the ξ_n are independent, and also $E\hat{X}_{n,j} = 0$ for all n, j. Additionally,

$$n \operatorname{Var} X_{n,j} = c_1^2 + \dots + c_m^2$$

$$+ 2 \sum_{1\le\mu<\nu\le m} c_\mu c_\nu \frac{\left(E\, I_{[\xi_1 \le \min\{u_n^{(\mu)}, u_n^{(\nu)}\}]} - p_n^{(\mu)} p_n^{(\nu)}\right)}{\sqrt{p_n^{(\mu)} q_n^{(\mu)} p_n^{(\nu)} q_n^{(\nu)}}} .$$

Since $p_n^{(\mu)} = F\left(u_n^{(\mu)}\right) \sim \frac{k_n^{(\mu)}}{n}$, we may suppose for all sufficiently large n that $u_n^{(\mu)} \le u_n^{(\nu)}$ for $\mu \le \nu$, and thus $E\, I_{[\xi_1 \le \min\{u_n^{(\mu)}, u_n^{(\nu)}\}]} = p_n^{(\mu)}$. Therefore

$$n \operatorname{Var} \hat{X}_{n,j} = c_1^2 + \dots + c_m^2 + 2 \sum_{1\le\mu<\nu\le m} c_\mu c_\nu \frac{p_n^{(\mu)} p_n^{(\nu)}}{\sqrt{p_n^{(\mu)} q_n^{(\mu)} p_n^{(\nu)} q_n^{(\nu)}}}$$

$$(3.1.5) \qquad \to c_1^2 + \dots + c_m^2 + 2 \sum_{1\le\mu<\nu\le m} c_\mu c_\nu\, \sigma_{\mu,\nu}$$

as $n \to \infty$, where

$$\sigma_{\mu,\nu} = \lim_n \sqrt{\frac{p_n^{(\mu)} q_n^{(\nu)}}{q_n^{(\mu)} p_n^{(\nu)}}} ,$$

$1 \le \mu < \nu \le m$. Also, since

$$n p_n^{(\mu)} q_n^{(\mu)} \sim \frac{k_n^{(\mu)}\left(n - k_n^{(\mu)}\right)}{n} \to \infty,$$

it is clear that the r.v.'s $\hat{X}_{n,1}, \dots, \hat{X}_{n,n}$ are uniformly bounded in $j = 1, \dots, n$ by a numerical quantity tending to zero as $n \to \infty$, so that the array $\{\hat{X}_{n,j}\}$ satisfies the Lindeberg condition

$$\lim_n \sum_{j=1}^{n} E\left(\hat{X}_{n,j}^2\, I_{[|\hat{X}_{n,j}| > \varepsilon]}\right) = 0 \quad \text{for every } \varepsilon > 0.$$

Thus by the usual central limit theorem for row sums from triangular arrays (Loève (1963, p. 295)), we have that $c_1 X_n^{(1)} + \dots + c_m X_n^{(m)} = \sum_{j=1}^{n} \hat{X}_{n,j}$ is asymptotically normal with mean zero and variance the limiting quantity (3.1.5). That is, equivalently,

$$\mathcal{L}(c_1 X_n^{(1)} + \ldots + c_m X_n^{(m)}) \to \mathcal{L}(c_1 X^{(1)} + \ldots + c_m X^{(m)}),$$

where the $X^{(\mu)}$ are standard normal r.v.'s with correlations

$$E\, X^{(\mu)} X^{(\nu)} = \sigma_{\mu,\nu} \,,$$

$1 \le \mu < \nu \le m$. Hence by the Cramér-Wold device,

$$(3.1.6) \quad \mathcal{L}\left[\left(X_n^{(1)}, \ldots, X_n^{(m)}\right)\right] \to \mathcal{L}\left[\left(X^{(1)}, \ldots, X^{(m)}\right)\right],$$

where the random vector $(X^{(1)}, \ldots, X^{(m)})$ has the zero-mean m-dimensional normal distribution Φ_Σ having as covariance the $m \times m$ symmetric matrix Σ whose (μ,ν)-th component is $\sigma_{\mu,\nu}$, $1 \le \mu < \nu \le m$, and which has units on its principal diagonal. Thus combining (3.1.3), (3.1.4), and (3.1.6) we obtain

$$P\left[\frac{\xi^{(n)}_{k_n^{(1)}} - b_n^{(m)}}{a_n^{(1)}} \le x_1, \ldots, \frac{\xi^{(n)}_{k_n^{(m)}} - b_n^{(m)}}{a_n^{(m)}} \le x_m\right] \to \Phi_\Sigma\left(u^{(1)}(x_1), \ldots, u^{(m)}(x_m)\right).$$

Finally let us evaluate the components

$$\sigma_{\mu,\nu} = \lim_n \sqrt{\frac{p_n^{(\mu)} q_n^{(\nu)}}{q_n^{(\mu)} p_n^{(\nu)}}} \,.$$

(i) If $\frac{k_n^{(\mu)}}{n} \to 0$ and $\frac{k_n^{(\nu)}}{n} \to 0$, that is, $1 \le \mu < \nu \le m_1$, then

$$\sigma_{\mu,\nu} = \lim_n \sqrt{\frac{k_n^{(\mu)}}{k_n^{(\nu)}}}\,,$$ which we assume to exist.

(ii) If $\frac{k_n^{(\mu)}}{n} \to 0$ and $\frac{k_n^{(\nu)}}{n} \to \lambda$, $0 < \lambda \le 1$, then clearly $\sigma_{\mu,\nu} = 0$, since $p_n^{(\mu)} \to 0$ and $p_n^{(\nu)} \to \lambda$.

(iii) If $\frac{k_n^{(\mu)}}{n} \to \lambda_\mu$ and $\frac{k_n^{(\mu)}}{n} \to \lambda_\nu$, $0 < \lambda_\mu \le \lambda_\nu < 1$, then

$$\sigma_{\mu,\nu} = \sqrt{\frac{\lambda_\mu(1-\lambda_\nu)}{\lambda_\nu(1-\lambda_\mu)}} \,.$$

(iv) If $\frac{k_n^{(\mu)}}{n} \to \lambda_\mu$, $0 < \lambda_\mu < 1$, and $\frac{k_n^{(\nu)}}{n} \to 1$, then $\sigma_{\mu,\nu} = 0$, since $q_n^{(\nu)} \to 0$.

(v) If $\frac{k_n^{(\mu)}}{n} \to 1$ and $\frac{k_n^{(\nu)}}{n} \to 1$, $\mu < \nu$, then $\sigma_{\mu,\nu} = \lim_n \sqrt{\frac{q_n^{(\nu)}}{q_n^{(\mu)}}} =$ $\lim_n \sqrt{\frac{n-k_n^{(\nu)}}{n-k_n^{(\mu)}}}$, assumed to exist.

Hence from (i)-(v) the components of the covariance matrix Σ are as claimed in the statement of the theorem. It is worth noting that the components do not depend upon $x_1, \ldots, x_m$.

For a particular type of the intermediate rank sequences considered, we have the following result as a special case of the previous theorem.

Theorem 3.1.2. Suppose k_n is a given intermediate rank sequence and ξ_n is i.i.d. with marginal d.f. F belonging to the domain of attraction for smallest terms of the law $G^{(t)}(x) = \Phi(u^{(t)}(x))$ for each of the intermediate rank sequences $k_n^{(t)}$ defined by

$$k_n^{(t)} = \max\{1, [tk_n]\},$$

for $0 < t \le 1$; that is, there are real sequences $a_n^{(t)} > 0$, $b_n^{(t)}$ such that

$$\left[\frac{\xi_{k_n^{(t)}}^{(n)} - b_n^{(t)}}{a_n^{(t)}} \le x\right] \to G^{(t)}(x)$$

for all real x, for each $0 < t \le 1$. Then for each choice of $0 < t_1 < \ldots < t_m \le 1$,

$$P\left[\frac{\xi^{(n)}_{k_n^{(t_1)}} - b_n^{(t_1)}}{a_n^{(t_1)}} \le x_1, \ldots, \frac{\xi^{(n)}_{k_n^{(t_m)}} - b_n^{(t_m)}}{a_n^{(t_m)}} \le x_m\right]$$

$$\to \Phi_\Sigma\left(u^{(t_1)}(x_1), \ldots, u^{(t_m)}(x_m)\right)$$

for all real $x_1, \ldots, x_m$, where Φ_Σ is the zero-mean m-dimensional normal law with covariance matrix Σ whose (μ,ν)-th component is $\sqrt{\frac{t_\mu}{t_\nu}}$, $1 \le \mu \le \nu \le m$. Thus if a sequence of stochastic processes $\{Y_n(t),\ 0 \le t \le 1\}_{n \ge 1}$ is defined by

$$Y_n(t) = \begin{cases} \sqrt{t}\ \dfrac{\xi^{(n)}_{k_n^{(t)}} - b_n^{(t)}}{a_n^{(t)}}, & 0 < t \le 1 \\ 0, & t = 0 \end{cases},$$

then for any $0 \le t_1 < \ldots \le t_m \le 1$,

$$\left[\left(Y_n(t_1), \ldots, Y_n(t_m)\right)\right] \to \Phi_{\Sigma'},$$

where $\Phi_{\Sigma'}$ is the zero-mean m-dimensional normal law with covariance matrix Σ' whose (μ,ν)-th component is t_μ, $1 \le \mu \le \nu \le m$; that is, the finite-dimensional distributions of Y_n converge to those of the (standard) Weiner process on the unit interval. (See Section 5.2.)

It is evident that the method used in the proof of Theorem 3.1.1 does not specifically require the attraction of the marginal d.f. F to the one-dimensional limit law for each of the increasing rank sequences being considered, but only that appropriate u_n levels can be chosen, that is, to satisfy (1.3.5), (1.3.10), and (1.3.11). Thus upon rephrasing the proof of Theorem 3.1.1 we can show the following result,

which includes as a particular case the formulation and solution of the multi-level exceedance problem relevant to intermediate and central rank sequences.

Theorem 3.1.3. Let m_1, m_2, and m_3 be nonnegative integers and $m = m_1 + m_2 + m_3$. Let $k_n^{(1)}, \ldots, k_n^{(m)}$ be m increasing rank sequences such that for some integer $N \geq 1$, $k_n^{(1)} \leq \ldots \leq k_n^{(m_1+m_2)}$ and $k_n^{(m_1+m_2+1)} \geq \ldots \geq k_n^{(m)}$, for all $n \geq N$, and suppose

$$\lim_n \frac{k_n^{(\mu)}}{n} = 0 \quad , \qquad \mu = 1, \ldots, m_1 ,$$

$$\lim_n \frac{k_n^{(m_1+\mu)}}{n} = \lambda_{m_1+\mu} , \qquad \mu = 1, \ldots, m_2 ,$$

where

$$0 < \lambda_{m_1+1} \leq \ldots \leq \lambda_{m_1+m_2} < 1,$$

and

$$\lim_n \frac{k_n^{(m_1+m_2+\mu)}}{n} = 0, \qquad \mu = 1, \ldots, m_3 .$$

Suppose also that

$$\lambda_{\mu,\nu} = \lim_n \frac{k_n^{(\mu)}}{k_n^{(\nu)}}$$

exists for each $1 \leq \mu \leq \nu \leq m_1$, and that

$$\lambda_{m_1+m_2+\mu, m_1+m_2+\nu} = \lim_n \frac{k_n^{(m_1+m_2+\nu)}}{k_n^{(m_1+m_2+\mu)}}$$

exists for each $1 \leq \mu \leq \nu \leq m_3$. Let ξ_n be i.i.d. with marginal d.f. F. Let $u^{(1)}, \ldots, u^{(m)}$ be real numbers, and suppose for $\mu = 1, \ldots, m_1$, the level $u_n^{(\mu)}$ is such that

$$1 - F(u_n^{(\mu)}) = \frac{k_n^{(\mu)}}{n} - \frac{u^{(\mu)}\sqrt{k_n^{(\mu)}}}{n} + o\left(\frac{\sqrt{k_n^{(\mu)}}}{n}\right);$$

suppose for $\mu = 1, \ldots, m_2$, the level $u_n^{(m_1+\mu)}$ is such that

$$1 - F(u_n^{(\mu)}) = \frac{k_n^{(m_1+\mu)}}{n} - \frac{u^{(m_1+\mu)}\sqrt{\lambda_{m_1+\mu}(1-\lambda_{m_1+\mu})}}{n} + o\left(\frac{1}{\sqrt{n}}\right);$$

also suppose for $\mu = 1, \ldots, m_3$, $u_n^{(m_1+m_2+\mu)}$ is such that

$$F\left(u_n^{(m_1+m_2+\mu)}\right) = \frac{k_n^{(m_1+m_2+\mu)}}{n} + \frac{u^{(m_1+m_2+\mu)}\sqrt{k_n^{(m_1+m_2+\mu)}}}{n} + o\left(\frac{\sqrt{k_n^{(m_1+m_2+\mu)}}}{n}\right).$$

Now let $W_n^{(\mu)} = \sum_{i=1}^{n} I_{[\xi_i > u_n^{(\mu)}]}$ be the number of exceedances of $u_n^{(\mu)}$ by $\xi_1, \ldots, \xi_n$, for $\mu = 1, \ldots, m_1+m_2$, and let $T_n^{(m_1+m_2+\mu)} = \sum_{i=1}^{n} I_{[\xi_i \le u_n^{(m_1+m_2+\mu)}]}$ be the number of $\xi_1,\ldots,\xi_n$ which do not exceed $u_n^{(m_1+m_2+\mu)}$, for $\mu = 1, \ldots, m_3$. Then

$$\begin{aligned}
P\Big(&W_n^{(\mu)} < k_n^{(\mu)}, \quad \mu = 1, \ldots, m_1; \\
&W_n^{(m_1+\mu)} < k_n^{(m_1+\mu)}, \quad \mu = 1, \ldots, m_2; \\
&T_n^{(m_1+m_2+\mu)} \ge k_n^{(m_1+m_2+\mu)}, \quad \mu = 1, \ldots, m_3\Big) \\
&\to \Phi_{\Sigma_1}(u^{(1)}, \ldots, u^{(m_1)}) \cdot \Phi_{\Sigma_2}(u^{(m_1+1)}, \ldots, u^{(m_1+m_2)}) \\
&\quad \cdot \Phi_{\Sigma_3}(u^{(m_1+m_2+1)}, \ldots, u^{(m)})
\end{aligned}$$

as $n \to \infty$, where for $j = 1,2,3$, Φ_{Σ_j} is the zero-mean m_j-dimensional normal law with (symmetric) covariance matrix Σ_j, respectively given by

$$\Sigma_1 = (\lambda_{\mu,\nu}^{1/2})_{1\le\mu\le\nu\le m_1},$$

$$\Sigma_2 = \left(\left(\frac{\lambda_{m_1+\mu}\left(1-\lambda_{m_1+\nu}\right)}{\lambda_{m_1+\nu}\left(1-\lambda_{m_1+\mu}\right)}\right)^{\frac{1}{2}}\right)_{1\le\mu\le\nu\le m_2},$$

and

$$\Sigma_3 = \left(\lambda^{\frac{1}{2}}_{m_1+m_2+\mu,m_1+m_2+\nu}\right)_{1\le\mu\le\nu\le m_3}.$$

We conclude this section by stating, in our notation, the following result of Chen (1966, Lemma 9), which gives the limiting joint distribution of fixed rank and increasing rank order statistics, when the one-dimensional limiting distributions exist. As is evident these are asymptotically mutually independent, and combining this result with Theorem 3.1.1 we have the asymptotic mutual independence of order statistics of the five rank classes fixed largest, largest intermediate, central, smallest intermediate, and fixed smallest. Moreover, we have the asymptotic mutual independence of the numbers of exceedances of the levels u_n defined by (1.2.10), (1.3.5), and (1.3.11), when the marginal d.f. F of the ξ_n sequence is such that these levels can be so chosen.

Theorem 3.1.4. Let m_1, m_2, and m_3 be nonnegative integers and $m = m_1 + m_2 + m_3$. For each $\mu = 1, \ldots ,m$ and each $n \ge 1$ let $W_n^{(\mu)}$ be a sum of n i.i.d. indicator r.v.'s such that

$$W_n^{(1)} \le \ldots \le W_n^{(m_1+m_2)} \le n - W_n^{(m_1+m_2+1)} \le \ldots \le n - W_n^{(m)} .$$

Suppose, as $n \to \infty$,

$$P\left(W_n^{(1)} = k_1, \ldots ,W_n^{(m_1)} = k_{m_1}\right) \to P\left(I_1 = k_1, \ldots ,I_{m_1} = k_{m_1}\right)$$

for any nonnegative integers $k_1, \ldots ,k_{m_1}$, where $\left(I_1, \ldots ,I_{m_1}\right)$ is an independent increment Poisson vector; suppose

$$P\left(\frac{W_n^{(m_1+1)}-EW_n^{(m_1+1)}}{\mathrm{Var}^{\frac{1}{2}}W_n^{(m_1+1)}} \le u^{(m_1+1)}, \dots, \frac{W_n^{(m_1+m_2)}-EW_n^{(m_1+m_2)}}{\mathrm{Var}^{\frac{1}{2}}W_n^{(m_1+m_2)}} \le u^{(m_1+m_2)}\right)$$

$$\to \Phi_\Sigma\left(u^{(m_1+1)}, \dots, u^{(m_1+m_2)}\right)$$

for all real $u^{(m_1+1)}, \dots, u^{(m_1+m_2)}$, where Φ_Σ is an m_2-dimensional normal law; also suppose that

$$P\left(W_n^{(m_1+m_2+1)} = k_{m_1+m_2+1}, \dots, W_n^{(m)} = k_m\right)$$

$$\to P\left(J_{m_1+m_2+1} = k_{m_1+m_2+1}, \dots, J_m = k_m\right)$$

for any m_3 nonnegative integers $\left(k_{m_1+m_2+1}, \dots, k_m\right)$, where $\left(J_{m_1+m_2+1}, \dots, J_m\right)$ is a reverse independent increment Poisson vector. Then

$$P\left(W_n^{(1)} = k_1, \dots, W_n^{(m_1)} = k_{m_1} ;\right.$$

$$\frac{W_n^{(m_1+1)}-EW_n^{(m_1+1)}}{\mathrm{Var}^{\frac{1}{2}}W_n^{(m_1+1)}} \le u^{(m_1+1)}, \dots, \frac{W_n^{(m_1+m_2)}-EW_n^{(m_1+m_2)}}{\mathrm{Var}^{\frac{1}{2}}W_n^{(m_1+m_2)}} \le u^{(m_1+m_2)};$$

$$\left. W_n^{(m_1+m_2+1)} = k_{m_1+m_2+1}, \dots, W_n^{(m)} = k_m\right)$$

$$\to P\left(I_1=k_1, \dots, I_{m_1}=k_{m_1}\right) \cdot \Phi_\Sigma\left(u^{(m_1+1)}, \dots, u^{(m_1+m_2)}\right)$$

$$\cdot P\left(J_{m_1+m_2+1}=k_{m_1+m_2+1}, \dots, J_m=k_m\right)$$

as $n \to \infty$, for all nonnegative integers $k_1, \dots, k_{m_1}, k_{m_1+m_2+1}, \dots, k_m$, and for all real $u^{(m_1+1)}, \dots, u^{(m_1+m_2)}$.

3.2. The Stationary Case.

In this section we give one set of conditions under which results of the previous section are valid for stationary, dependent sequences ξ_n . These conditions are similar to those used in Section 2.3 for the one-dimensional limiting distribution and single-level exceedance problems, though somewhat more complicated due to the consideration of several rank sequences simultaneously, and therefore we deal exclusively with *intermediate order statistics.*

To briefly describe the results to be obtained, we will give conditions under which the random vector $\left(\xi^{(n)}_{k_n^{(1)}}, \ldots, \xi^{(n)}_{k_n^{(m)}}\right)$, normalized, has the same asymptotic distribution as it would if the stationary sequence ξ_n were actually independent, where we assume the random vector $\left(\hat{\xi}^{(n)}_{k_n^{(1)}}, \ldots, \hat{\xi}^{(n)}_{k_n^{(m)}}\right)$ indeed has an asymptotic distribution, $\hat{\xi}_n$ being the independent sequence associated with ξ_n . In fact, we will find it convenient to work directly with the more general multi-level exceedance problem developed for i.i.d. sequences in Theorem 3.1.3, where we restrict consideration to u_n levels determined by intermediate rank sequences.

Throughout this section we assume that ξ_n is stationary with marginal d.f. F. Let m_1 and m_2 be nonnegative integers and $m = m_1 + m_2$. Suppose that $k_n^{(1)}, \ldots, k_n^{(m)}$ are m given intermediate rank sequences such that

$$(3.2.1)\quad k_n^{(1)} \le \ldots \le k_n^{(m_1)} \quad \text{and} \quad k_n^{(m_1+1)} \ge \ldots \ge k_n^{(m)}$$

for all sufficiently large n. For our purposes we define

$$\check{k}_n = \max\{k_n^{(\mu)},\ 1 \le \mu \le m\}$$

and

$$\hat{k}_n = \min\{k_n^{(\mu)},\ 1 \le \mu \le m\}.$$

As in the one-dimensional problem considered in Chapter II, we construct the following partition of the first n positive integers, for each large n: Two given sequences of integers $\alpha_n \to \infty$ and $\beta_n \to \infty$ such that $\alpha_n = o(n)$ and $\beta_n = o(\alpha_n)$ determine intervals $I_1^{(n)}, \ldots, I_{2\ell_n}^{(n)}$ of respective lengths $\alpha_n, \beta_n, \ldots, \alpha_n, \gamma_n$, where $n = (\alpha_n + \beta_n)\ell_n + \gamma_n$ and $0 \le \gamma_n < \alpha_n + 2\beta_n$, and moreover, $\ell_n \sim n\alpha_n^{-1}$. Now let $u_n^{(1)}, \ldots, u_n^{(m)}$ be m real sequences,

$$W_{I_i^{(n)}}^{(\mu)} = \sum_{j \in I_i^{(n)}} I_{[\xi_j > u_n^{(\mu)}]}\ ; \qquad 1 \le i \le 2\ell_n\ , \quad 1 \le \mu \le m_1\ ,$$

and

$$T_{I_i^{(n)}}^{(\mu)} = \sum_{j \in I_i^{(n)}} I_{[\xi_j \le u_n^{(\mu)}]}\ ; \qquad 1 \le i \le 2\ell_n\ , \quad m_1+1 \le \mu \le m.$$

Also define

$$\sigma_n^2(\mu) = \begin{cases} \operatorname{Var} W_{I_1^{(n)}}^{(\mu)}\ , & 1 \le \mu \le m_1 \\ \operatorname{Var} T_{I_1^{(n)}}^{(\mu)}\ , & m_1+1 \le \mu \le m \end{cases}.$$

We consider the following assumption:

J: There is a sequence of integers $\alpha_n \to \infty$ and $\alpha_n = o(\sqrt{k_n})$ such that

$$(3.2.2)\quad \alpha_n^{-1}\sum_{j=1}^{\alpha_n-1}(\alpha_n-j)\Big\{P\big(\xi_1>u_n^{(\mu)},\ \xi_{1+j}>u_n^{(\nu)}\big) + P\big(\xi_1>u_n^{(\nu)},\ \xi_{1+j}>u_n^{(\mu)}\big) - 2\Big[1-F\big(u_n^{(\mu)}\big)\Big]\Big[1-F\big(u_n^{(\nu)}\big)\Big] = o\left(\frac{k_n^{(\mu)}}{n}\right)$$

for each $1\le\mu\le\nu\le m_1$,

$$(3.2.3)\quad \alpha_n^{-1}\sum_{j=1}^{\alpha_n-1}(\alpha_n-j)\Big\{P\big(\xi_1\le u_n^{(\mu)},\ \xi_{1+j}\le u_n^{(\nu)}\big) + P\big(\xi_1\le u_n^{(\nu)},\ \xi_{1+j}\le u_n^{(\mu)}\big) - 2F\big(u_n^{(\mu)}\big)F\big(u_n^{(\nu)}\big)\Big\} = o\left(\frac{k_n^{(\nu)}}{n}\right)$$

for each $m_1+1\le\mu\le\nu\le m$, and

$$(3.2.4)\quad \alpha_n^{-1}\sum_{j=1}^{\alpha_n-1}(\alpha_n-j)\Big\{P\big(\xi_1>u_n^{(\mu)},\ \xi_{1+j}\le u_n^{(\nu)}\big) + P\big(\xi_1\le u_n^{(\nu)},\ \xi_{1+j}>u_n^{(\mu)}\big) - 2\Big[1-F\big(u_n^{(\mu)}\big)\Big]F\big(u_n^{(\nu)}\big)\Big\} = o\left(\frac{\sqrt{k_n^{(\mu)}k_n^{(\nu)}}}{n}\right)$$

for each $1\le\mu\le m_1<\nu\le m$; and,

For some α_n for which (3.2.2)-(3.2.4) hold, there is a sequence of integers $\beta_n\to\infty$, $\beta_n=o(\alpha_n)$, and $\beta_n\check{k}_n=o(n)$, such that

$$(3.2.5)\qquad \frac{n}{\alpha_n}g_n\to 0,$$

where

$$g_n=\sup_{1\le s<2\ell_n-1}\ \sup_{\substack{A\in\mathcal{F}_{n,s}\\ B\in\mathcal{A}_{n,s+2}}}|P(AB)-P(A)P(B)|,$$

$\mathcal{F}_{n,s}$ being the σ-field generated by the r.v.'s

$$\left\{\sum_{\mu=1}^{m_1} W^{(\mu)}_{I_i^{(n)}} + \sum_{\mu=m_1+1}^{m} T^{(\mu)}_{I_i^{(n)}}\ , \quad i = 1, \dots ,s\right\}$$

and $\mathcal{A}_{n,s+2}$ the σ-field generated by the r.v.'s

$$\left\{\sum_{\mu=1}^{m_1} W^{(\mu)}_{I_i^{(n)}} + \sum_{\mu=m_1+1}^{m} T^{(\mu)}_{I_i^{(n)}}\ , \quad i = s+2, \dots ,2\ell_n\right\}.$$

Corresponding to (2.3.4) in the one-dimensional problem, a condition stronger than (3.2.5) in that more events must be considered, is

(3.2.6) $$\frac{n}{\alpha_n} h_n(\beta_n) \to 0,$$

where

$$h_n(\ell) = \sup_{1\le s<n-\ell} \ \sup_{\substack{A\in\mathcal{F}'_{n,s} \\ B\in\mathcal{A}'_{n,s+\ell+1}}} |P(AB) - P(A)P(B)|,$$

$\mathcal{F}'_{n,s}$ being the σ-field generated by the collection

$$\left\{I_{[\xi_i>u_n^{(\mu)}]}\ , \quad \mu = 1, \dots ,m_1\ , \quad i = 1, \dots ,s\ ;\right.$$

$$\left. I_{[\xi_i\le u_n^{(\mu)}]}\ , \quad \mu = m_1+1, \dots ,m, \quad i = 1, \dots ,s\right\}$$

and $\mathcal{A}'_{n,s+\ell+1}$ the σ-field generated by

$$\left\{I_{[\xi_i>u_n^{(\mu)}]}\ , \quad \mu = 1, \dots ,m_1\ , \quad i = s+\ell+1, \dots ,n\ ;\right.$$

$$\left. I_{[\xi_i\le u_n^{(\mu)}]}\ , \quad \mu = m_1+1, \dots ,m, \quad i = s+\ell+1, \dots ,n\right\}.$$

Suppose that for large n we have $u_n^{(\mu)} \ge u_n^{(\nu)}$ if $1 \le \mu < \nu \le m$. Define intervals on the real line by

$$G_{n,o} = \left(u_n^{(1)}, \infty\right),$$

$$G_{n,\mu} = (u_n^{(\mu+1)}, u_n^{(\mu)}], \qquad 1 \le \mu < m,$$

and

$$G_{n,m} = (-\infty, u_n^{(m)}].$$

Then, equivalently, $\mathcal{F}'_{n,s}$ is generated by the r.v.'s

$$\{I_{[\xi_i \in G_{n,\mu}]}, \quad \mu = 0, 1, \dots, m, \quad i = 1, \dots, s\}$$

and $\mathcal{A}'_{n,s+\beta_n+1}$ by

$$\{I_{[\xi_i \in G_{n,\mu}]}, \quad \mu = 0, 1, \dots, m, \quad i = s+\beta_n+1, \dots, n\}.$$

Now for fixed n, s let $\mu' = \{\mu_1, \dots, \mu_s\}$ be a finite sequence of length s and $\mu'' = \{\mu_{s+\beta_n+1}, \dots, \mu_n\}$ a sequence of length $n - \beta_n - s - 1$, both of whose components are the integers $0, 1, \dots, m$. Define events

$$A_{\mu'} = \{\xi_k \in G_{n,\mu_k}, \quad k = 1, \dots, s\}$$

and

$$B_{\mu''} = \{\xi_k \in G_{n,\mu_k}, \quad k = s+\beta_n+1, \dots, n\}.$$

It is not difficult to see that each $A \in \mathcal{F}'_{n,s}$ is some disjoint union of at most $(m + 1)^s$ events each of the form $A_{\mu'}$, and similarly each $B \in \mathcal{A}'_{n,s+\beta_n+1}$ is a disjoint union of at most $(m + 1)^{n-s-\beta_n}$ events of the form $B_{\mu''}$. Hence a sufficient condition for (3.2.6) is

$$(3.2.7) \qquad \sup_{1\le s<n-\beta_n} \sup_{\mu',\mu''} |P(A_{\mu'},B_{\mu''}) - P(A_{\mu'})P(B_{\mu''})| = o\left(\frac{\alpha_n}{n} (m + 1)^n\right).$$

However we will work directly with the assumption J as stated in deriving our main result below.

<u>Theorem 3.2.1.</u> Let ξ_n be a stationary sequence with marginal d.f.

F. Let $k_n^{(1)}, \ldots, k_n^{(m)}$ be $m = m_1 + m_2$ intermediate rank sequences satisfying (3.2.1) and assume that $\lambda_{\mu,\nu} = \lim_n \frac{k_n^{(\mu)}}{k_n^{(\nu)}}$ exists for each $1 \le \mu \le \nu \le m_1$, and that $\lambda_{\mu,\nu} = \lim_n \frac{k_n^{(\nu)}}{k_n^{(\mu)}}$ exists for each $m_1+1 \le \mu \le \nu \le m$. Given real numbers $u^{(1)}, \ldots, u^{(m)}$, suppose for $\mu = 1, \ldots, m_1$, the level $u_n^{(\mu)}$ is such that

$$1 - F\left(u_n^{(\mu)}\right) = \frac{k_n^{(\mu)}}{n} - \frac{u_n^{(\mu)}\sqrt{k_n^{(\mu)}}}{n} + o\left(\frac{\sqrt{k_n^{(\mu)}}}{n}\right) .$$

and for $\mu = m_1+1, \ldots, m$, the level $u_n^{(\mu)}$ is such that

$$F\left(u_n^{(\mu)}\right) = \frac{k_n^{(\mu)}}{n} + \frac{u^{(\mu)}\sqrt{k_n^{(\mu)}}}{n} + o\left(\frac{\sqrt{k_n^{(\mu)}}}{n}\right) .$$

Let $W_n^{(\mu)} = \sum_{i=1}^{n} I_{[\xi_i > u_n^{(\mu)}]}$ be the number of exceedances of $u_n^{(\mu)}$ by $\xi_1, \ldots, \xi_n$, for $1 \le \mu \le m_1$, and let $T_n^{(\mu)} = \sum_{i=1}^{n} I_{[\xi_i \le u_n^{(\mu)}]}$ be the number of $\xi_1, \ldots, \xi_n$ which do not exceed $u_n^{(\mu)}$, for $m_1+1 \le \mu \le m$. Then if the assumption J holds for the set of sequences $\left(u_n^{(1)}, \ldots, u_n^{(m)}\right)$, we have

$$P\left(W_n^{(\mu)} < k_n^{(\mu)}, \mu = 1, \ldots, m_1 ; \quad T_n^{(\mu)} \ge k_n^{(\mu)}, \mu = m_1+1, \ldots, m\right)$$
$$\to \Phi_{\Sigma_1}\left(u^{(1)}, \ldots, u^{(m_1)}\right) \cdot \Phi_{\Sigma_2}\left(u^{(m_1+1)}, \ldots, u^{(m)}\right)$$

as $n \to \infty$, where for $j = 1,2$, Φ_{Σ_j} is the zero-mean m_j-dimensional normal law with covariance matrix Σ_j given by, respectively,

(3.2.8) $$\Sigma_1 = \left(\lambda_{\mu,\nu}^{1/2}\right)_{1 \le \mu \le \nu \le m_1}$$

and

(3.2.9) $$\Sigma_2 = \left(\lambda^{\frac{1}{2}}_{m_1+\mu,m_1+\nu}\right)_{1\le\mu\le\nu\le m_2} .$$

Proof: Let α_n and β_n be given sequences for which the requirements of assumption J are satisfied. We have

$$P\left(W_n^{(\mu)} < k_n^{(\mu)},\ \mu = 1, \dots ,m_1 ;\ T_n^{(\mu)} \ge k_n^{(\mu)},\ \mu = m_1+1, \dots ,m\right)$$
$$= P\left(X_n^{(\mu)} < v_n^{(\mu)},\ \mu = 1, \dots ,m_1 ;\ X_n^{(\mu)} \ge v_n^{(\mu)},\ \mu = m_1+1, \dots ,m\right)$$

where

$$X_n^{(\mu)} = \begin{cases} \dfrac{W_n^{(\mu)} - np_n^{(\mu)}}{\sqrt{\ell_n}\ \sigma_n(\mu)}, & \mu = 1, \dots ,m_1 \\[2ex] \dfrac{T_n^{(\mu)} - np_n^{(\mu)}}{\sqrt{\ell_n}\ \sigma_n(\mu)}, & \mu = m_1+1, \dots ,m \end{cases},$$

$$v_n^{(\mu)} = \frac{k_n^{(\mu)} - np_n^{(\mu)}}{\sqrt{\ell_n}\ \sigma_n(\mu)}, \qquad \mu = 1, \dots ,m,$$

and

$$p_n^{(\mu)} = \begin{cases} 1 - F\left(u_n^{(\mu)}\right), & \mu = 1, \dots ,m_1 \\ F\left(u_n^{(\mu)}\right), & \mu = m_1+1, \dots ,m \end{cases}.$$

For each $\mu = 1, \dots ,m$ we have $p_n^{(\mu)} \to 0$ and

(3.2.10) $$np_n^{(\mu)} \sim k_n^{(\mu)} .$$

Hence by stationarity and (3.2.2), for $\mu = 1, \dots ,m_1$,

$$\sigma_n^2(\mu) = \alpha_n p_n^{(\mu)}\left(1 - p_n^{(\mu)}\right) + 2 \sum_{1\le i<j\le\alpha_n} \Big\{P\left(\xi_i > u_n^{(\mu)},\ \xi_j > u_n^{(\mu)}\right)$$
$$- \left[1 - F\left(u_n^{(\mu)}\right)\right]^2\Big\}$$

$$= \alpha_n p_n^{(\mu)}\left(1 - p_n^{(\mu)}\right) + 2\sum_{j=1}^{\alpha_n - 1}(\alpha_n - j)\left\{P\left(\xi_1 > u_n^{(\mu)},\ \xi_{1+j} > u_n^{(\mu)}\right) - \left[1 - F\left(u_n^{(\mu)}\right)\right]^2\right\}$$

$$\sim \frac{\alpha_n k_n^{(\mu)}}{n},$$

and similarly also by (3.2.3) we in fact have

$$\sigma_n^2(\mu) \sim \frac{\alpha_n k_n^{(\mu)}}{n} \qquad (3.2.11)$$

for each $\mu = 1, \ldots, m$. Therefore, from (3.2.10) and (3.2.11) it follows that

$$v_n^{(\mu)} \to \begin{cases} u^{(\mu)}, & \mu = 1, \ldots, m_1 \\ -u^{(\mu)}, & \mu = m_1+1, \ldots, m \end{cases} \qquad (3.2.12)$$

Let $c_1, \ldots, c_m$ be real numbers and write

$$c_1 X_n^{(1)} + \ldots + c_m X_n^{(m)} = \sum_{i=1}^{2\ell_n} \hat{X}_{n,i},$$

where

$$\hat{X}_{n,i} = \sum_{\mu=1}^{m_1} c_\mu \frac{W_{I_i^{(n)}}^{(\mu)} - E\,W_{I_i^{(n)}}^{(\mu)}}{\sqrt{\ell_n}\,\sigma_n(\mu)} + \sum_{\mu=m_1+1}^{m} c_\mu \frac{T_{I_i^{(n)}}^{(\mu)} - E\,T_{I_i^{(n)}}^{(\mu)}}{\sqrt{\ell_n}\,\sigma_n(\mu)}.$$

If we can show that

$$\lim_n \sum_{i \text{ odd}} E\hat{X}_{n,i}^2 = \sum_{\mu=1}^{m} c_\mu^2 + 2\sum_{1\le\mu<\nu\le m_1} c_\mu c_\nu \lambda_{\mu,\nu}^{\frac{1}{2}} + 2\sum_{m_1+1\le\mu<\nu\le m} c_\mu c_\nu \lambda_{\mu,\nu}^{\frac{1}{2}}, \qquad (3.2.13)$$

$$(3.2.14) \qquad \lim_{n} \sum_{i \text{ even}} E\hat{X}^2_{n,i} = 0,$$

and

$$(3.2.15) \qquad \lim_{n} \sum_{i=1}^{2\ell_n} E\left(\hat{X}^2_{n,i}\, I_{[|X_{n,i}|>\varepsilon]}\right) = 0 \quad \text{for all} \quad \varepsilon > 0,$$

then since the σ-fields $\mathcal{F}_{n,s}$ and $\mathcal{G}_{n,s}$ clearly are also generated by the r.v.'s $\hat{X}_{n,1}, \ldots, \hat{X}_{n,s}$ and $\hat{X}_{n,s+2}, \ldots, \hat{X}_{n,n}$, respectively, it will follow from Lemma 2.3.1 that $\sum_{i=1}^{2\ell_n} X_{n,i}$ is asymptotically normal with mean zero and variance the limiting quantity in (3.2.13). Then by the Cramér-Wold device and (3.2.12), the statement of the theorem will be immediate.

We first have

$$\begin{aligned} E\hat{X}^2_{n,i} &= \sum_{\mu=1}^{m} \frac{c_n^2}{\ell_n} \\ &+ 2 \sum_{1 \le \mu < \nu \le m_1} \frac{c_\mu c_\nu}{\ell_n \sigma_n(\mu) \sigma_n(\nu)} E\left\{\left(W^{(\mu)}_{I_1^{(n)}} - EW^{(\mu)}_{I_1^{(n)}}\right)\left(W^{(\nu)}_{I_1^{(n)}} - EW^{(\nu)}_{I_1^{(n)}}\right)\right\} \\ &+ 2 \sum_{m_1+1 \le \mu < \nu \le m} \frac{c_\mu c_\nu}{\ell_n \sigma_n(\mu) \sigma_n(\nu)} E\left\{\left(T^{(\mu)}_{I_1^{(n)}} - ET^{(\mu)}_{I_1^{(n)}}\right)\left(T^{(\nu)}_{I_1^{(n)}} - ET^{(\nu)}_{I_1^{(n)}}\right)\right\} \\ &+ 2 \sum_{1 \le \mu \le m_1 < \nu \le m} \frac{c_\mu c_\nu}{\ell_n \sigma_n(\mu) \sigma_n(\nu)} E\left\{\left(W^{(\mu)}_{I_1^{(n)}} - EW^{(\mu)}_{I_1^{(n)}}\right)\left(T^{(\nu)}_{I_1^{(n)}} - ET^{(\nu)}_{I_1^{(n)}}\right)\right\} \\ &= A_1^{(n)} + A_2^{(n)} + A_3^{(n)} + A_4^{(n)}, \end{aligned} \tag{3.2.16}$$

say. Now for large n, $u_n^{(\mu)} \ge u_n^{(\nu)}$ if $\mu \le \nu$, so that for fixed $1 \le \mu < \nu \le m_1$,

$$E\left\{\left(W^{(\mu)}_{I_1^{(n)}} - EW^{(\mu)}_{I_1^{(n)}}\right)\left(W^{(\nu)}_{I_1^{(n)}} - EW^{(\nu)}_{I_1^{(n)}}\right)\right\}$$

$$= E\left\{\left(\sum_{j=1}^{\alpha_n} I_{[\xi_j > u_n^{(\mu)}]} - \alpha_n p_n^{(\mu)}\right)\left(\sum_{k=1}^{\alpha_n} I_{[\xi_k > u_n^{(\nu)}]} - \alpha_n p_n^{(\nu)}\right)\right\}$$

$$= E\left\{\sum_{j=1}^{\alpha_n} I_{[\xi_j > u_n^{(\mu)}]} + \sum_{\substack{1\le j,k\le\alpha_n \\ j\ne k}} I_{[\xi_j > u_n^{(\mu)}, \xi_k > u_n^{(\nu)}]} - \alpha_n^2 p_n^{(\mu)} p_n^{(\nu)}\right\}$$

$$= \alpha_n p_n^{(\mu)}\left(1 - p_n^{(\nu)}\right)$$
$$+ \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\left\{P\left(\xi_1 > u_n^{(\mu)}, \xi_{1+j} > u_n^{(\nu)}\right) + P\left(\xi_1 > u_n^{(\nu)}, \xi_{1+j} > u_n^{(\mu)}\right)\right.$$
$$\left. - 2\, p_n^{(\mu)} p_n^{(\nu)}\right\}$$

$$\sim \frac{\alpha_n k_n^{(\mu)}}{n},$$

by stationarity, (3.2.2), and (3.2.10). Hence by (3.2.11),

$$\ell_n A_2^{(n)} \to 2 \sum_{1\le\mu<\nu\le m_1} c_\mu c_\nu\, \lambda_{\mu,\nu}^{1/2}\,. \tag{3.2.17}$$

Similarly, using (3.2.3), (3.2.10), and (3.2.11), we obtain

$$\ell_n A_3^{(n)} \to 2 \sum_{m_1+1\le\mu<\nu\le m} c_\mu c_\nu\, \lambda_{\mu,\nu}^{1/2}\,. \tag{3.2.18}$$

Also, for $1 \le \mu \le m_1 < \nu \le m$ and for large n,

$$E\left\{\left(W_{I_1^{(n)}}^{(\mu)} - E\,W_{I_1^{(n)}}^{(\mu)}\right)\left(T_{I_1^{(n)}}^{(\nu)} - E\,T_{I_1^{(n)}}^{(\nu)}\right)\right\}$$
$$= E\left\{\left(\sum_{j=1}^{\alpha_n} I_{[\xi_j > u_n^{(\mu)}]} - \alpha_n p_n^{(\mu)}\right)\left(\sum_{k=1}^{\alpha_n} I_{[\xi_k \le u_n^{(\nu)}]} - \alpha_n p_n^{(\nu)}\right)\right\}$$
$$= E\left\{\sum_{\substack{1\le j,k\le\alpha_n \\ j\ne k}} I_{[\xi_j > u_n^{(\mu)}, \xi_k \le u_n^{(\nu)}]} - \alpha_n^2 p_n^{(\mu)} p_n^{(\nu)}\right\}$$
$$= \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\left\{P\left(\xi_1 > u_n^{(\mu)}, \xi_{1+j} \le u_n^{(\nu)}\right)\right.$$
$$\left. + P\left(\xi_1 \le u_n^{(\nu)}, \xi_{1+j} > u_n^{(\mu)}\right) - 2 p_n^{(\mu)} p_n^{(\nu)}\right\} + \alpha_n p_n^{(\mu)} p_n^{(\nu)}$$

$$= \alpha_n p_n^{(\mu)} p_n^{(\nu)} + o\left(\frac{\alpha_n \sqrt{k_n^{(\mu)} k_n^{(\nu)}}}{n}\right)$$

by (3.2.4), and thus from (3.2.10) and (3.2.11),

$$\ell_n A_4^{(n)} \to 0. \tag{3.2.19}$$

Upon combining (3.2.16) - (3.2.19) and using stationarity we obtain (3.2.13).

Next we have, assuming $\ell_n > 1$,

$$\begin{aligned} E\hat{W}_{n,2}^2 &= \sum_{\mu=1}^{m_1} c_\mu^2 \frac{\operatorname{Var} W^{(\mu)}_{I_2^{(n)}}}{\ell_n \sigma_n^2(\mu)} + \sum_{\mu=m_1+1}^{m} c_\mu^2 \frac{\operatorname{Var} T^{(\mu)}_{I_2^{(n)}}}{\ell_n \sigma_n^2(\mu)} \\ &\quad + 2 \sum_{1 \le \mu < \nu \le m_1} \frac{c_\mu c_\nu}{\ell_n \sigma_n(\mu) \sigma_n(\mu)} E\left\{\left(W^{(\mu)}_{I_2^{(n)}} - E W^{(\mu)}_{I_2^{(n)}}\right)\right. \\ &\qquad \left.\cdot \left(W^{(\nu)}_{I_2^{(n)}} - E W^{(\nu)}_{I_2^{(n)}}\right)\right\} \\ &\quad + 2 \sum_{m_1+1 \le \mu < \nu \le m} \frac{c_\mu c_\nu}{\ell_n \sigma_n(\mu) \sigma_n(\mu)} E\left\{\left(T^{(\mu)}_{I_2^{(n)}} - E T^{(\mu)}_{I_2^{(n)}}\right)\right. \\ &\qquad \left.\cdot \left(T^{(\nu)}_{I_2^{(n)}} - E T^{(\nu)}_{I_2^{(n)}}\right)\right\} \\ &\quad + 2 \sum_{1 \le \mu \le m_1 < \nu \le m} \frac{c_\mu c_\nu}{\ell_n \sigma_n(\mu) \sigma_n(\nu)} E\left\{\left(W^{(\mu)}_{I_w^{(n)}} - E W^{(\mu)}_{I_2^{(n)}}\right)\right. \\ &\qquad \left.\cdot \left(T^{(\nu)}_{I_2^{(n)}} - E T^{(\nu)}_{I_2^{(n)}}\right)\right\} \end{aligned}$$

$$= B_1^{(n)} + B_2^{(n)} + B_3^{(n)} + B_4^{(n)} + B_5^{(n)}, \tag{3.2.20}$$

say. We may show that

$$\frac{\operatorname{Var} W^{(\mu)}_{I_2^{(n)}}}{\sigma_n^2(\mu)} \to 0, \qquad \mu = 1, \ldots, m_1,$$

and

$$\frac{\operatorname{Var} T^{(\mu)}_{I_2^{(n)}}}{\sigma_n^2(\mu)} \to 0, \qquad \mu = m_1+1, \ldots ,m,$$

using the same calculations as in the proof of Lemma 2.3.1, and we omit the details. Hence

$$B_1^{(n)} = o\left(\frac{1}{\ell_n}\right) \quad \text{and} \quad B_2^{(n)} = o\left(\frac{1}{\ell_n}\right).$$

Now, we have for $1 \le \mu < \nu \le m_1$, and n large,

$$E\left\{\left(W^{(\mu)}_{I_2^{(n)}} - E\,W^{(\mu)}_{I_2^{(n)}}\right)\left(W^{(\nu)}_{I_2^{(n)}} - E\,W^{(\nu)}_{I_2^{(n)}}\right)\right\}$$

$$= \sum_{1\le j,k\le \beta_n} P\left(\xi_j > u_n^{(\mu)},\ \xi_k > u_n^{(\nu)}\right) - \beta_n^2 p_n^{(\mu)} p_n^{(\nu)}$$

$$(3.2.21) \qquad = \beta_n p_n^{(\mu)} p_n^{(\nu)} + \sum_{j=1}^{\beta_n-1} (\beta_n - j)\left\{P\left(\xi_1 > u_n^{(\mu)},\ \xi_{1+j} > u_n^{(\nu)}\right)\right.$$

$$\left. + P\left(\xi_1 > u_n^{(\nu)},\ \xi_{1+j} > u_n^{(\mu)}\right) - 2p_n^{(\mu)} p_n^{(\nu)}\right\},$$

by stationarity. Then,

$$\left|\sum_{j=1}^{\beta_n-1} (\beta_n - j)\left\{P\left(\xi_1 > u_n^{(\mu)},\ \xi_{1+j} > u_n^{(\nu)}\right) + P\left(\xi_1 > u_n^{(\nu)},\ \xi_{1+j} > u_n^{(\mu)}\right)\right.\right.$$

$$\left.\left. - 2p_n^{(\mu)} p_n^{(\nu)}\right\}\right|$$

$$\le \sum_{j=1}^{\beta_n-1} j\left\{P\left(\xi_1 > u_n^{(\mu)},\ \xi_{\beta_n-j+1} > u_n^{(\nu)}\right) + P\left(\xi_1 > u_n^{(\nu)},\ \xi_{\beta_n-j+1} > u_n^{(\mu)}\right)\right.$$

$$= \sum_{j=\alpha_n-\beta_n+1}^{\alpha_n-1} \frac{j-(\alpha_n-\beta_n)}{j} \cdot j\left\{P\left(\xi_1 > u_n^{(\mu)},\ \xi_{\alpha_n-j+1} > u_n^{(\nu)}\right)\right.$$

$$\left. + P\left(\xi_1 > u_n^{(\nu)},\ \xi_{\alpha_n-j+1} > u_n^{(\mu)}\right)\right\}$$

$$\leq \frac{\beta_n}{\alpha_n - \beta_n} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\Big\{P\Big[\xi_1 > u_n^{(\mu)},\ \xi_{1+j} > u_n^{(\nu)}\Big]$$

$$+ P\Big[\xi_1 > u_n^{(\nu)},\ \xi_{1+j} > u_n^{(\mu)}\Big]\Big\}$$

$$= \frac{\beta_n}{\alpha_n} \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\Big\{P\Big[\xi_1 > u_n^{(\mu)},\ \xi_{1+j} > u_n^{(\nu)}\Big]$$

$$+ P\Big[\xi_1 > u_n^{(\nu)},\ \xi_{1+j} > u_n^{(\mu)}\Big] - 2\, p_n^{(\mu)} p_n^{(\nu)}\Big\}$$

$$+ \frac{\beta_n}{2(\alpha_n - \beta_n)} \alpha_n (\alpha_n - 1) p_n^{(\mu)} p_n^{(\nu)}$$

$$+ 2\left(\frac{\beta_n}{\alpha_n - \beta_n} - \frac{\beta_n}{\alpha_n}\right) \sum_{j=1}^{\alpha_n - 1} (\alpha_n - j)\Big\{P\Big[\xi_1 > u_n^{(\mu)},\ \xi_{1+j} > u_n^{(\nu)}\Big]$$

$$+ P\Big[\xi_1 > u_n^{(\nu)},\ \xi_{1+j} > u_n^{(\mu)}\Big] - 2\, p_n^{(\mu)} p_n^{(\nu)}\Big\}$$

(3.2.22) $$= C_1^{(n)} + C_2^{(n)} + C_3^{(n)},$$

say. By (3.2.4) we have that

(3.2.23) $$C_1^{(n)} = o\left(\frac{\beta_n k_n^{(\mu)}}{n}\right).$$

Also, it is easily seen that

(3.2.24) $$C_3^{(n)} = o\left(\frac{\beta_n k_n^{(\mu)}}{n}\right).$$

Moreover,

(3.2.25) $$C_2^{(n)} \sim \frac{\beta_n \alpha_n}{2} \cdot \frac{k_n^{(\mu)} k_n^{(\nu)}}{n^2} = o\left(\frac{\alpha_n k_n^{(\mu)}}{n}\right),$$

since $\beta_n \check{k}_n = o(n)$. Thus from (3.2.20) - (3.2.25) it follows that

$$B_3^{(n)} = o\left(\frac{1}{\ell_n}\right).$$

We may also show that $B_4^{(n)} = \left(\frac{1}{\ell_n}\right)$ and $B_5^{(n)} = \left(\frac{1}{\ell_n}\right)$ in virtually the same manner, and we omit these details. Therefore by stationarity,

$$(3.2.26) \qquad \sum_{\substack{i \text{ even} \\ i<2\ell_n}} E\hat{X}^2_{n,i} = (\ell_n - 1)\ E\hat{X}^2_{n,i} \to 0.$$

We now separately consider $E\hat{X}_{n,2\ell_n}$. We have

$$E\,\hat{X}^2_{n,2\ell_n} = B_1^{(n)} + B_2^{(n)} + B_3^{(n)} + B_4^{(n)} + B_5^{(n)}$$

as defined by (3.2.20), now replacing the interval $I_2^{(n)}$ by $I_{2\ell_n}^{(n)}$. It is clear that

$$\text{Var}\, W^{(\mu)}_{I^{(n)}_{2\ell_n}} \le (\alpha_n + 2\beta_n)p_n^{(\mu)}\left(1 - p_n^{(\mu)}\right), \qquad \mu = 1, \ldots, m_1 ,$$

and

$$\text{Var}\, T^{(\mu)}_{I^{(n)}_{2\ell_n}} \le (\alpha_n + 2\beta_n)p_n^{(\mu)}\left(1 - p_n^{(\mu)}\right), \qquad \mu = m_1+1, \ldots, m,$$

so that by (3.2.10) and (3.2.11) and $\ell_n \to \infty$,

$$B_1^{(n)} \to 0 \quad \text{and} \quad B_2^{(n)} \to 0.$$

Also, for $1 \le \mu < \nu \le m_1$,

$$E\left\{\left[W^{(\mu)}_{I^{(n)}_{2\ell_n}} - E\,W^{(\mu)}_{I^{(n)}_{2\ell_n}}\right]\left[W^{(\nu)}_{I^{(n)}_{2\ell_n}} - E\,W^{(\nu)}_{I^{(n)}_{2\ell_n}}\right]\right\}$$

$$\le \sum_{1\le j,k\le \alpha_n+2\beta_n} P\left[\xi_j > u_n^{(\mu)},\ \xi_k > u_n^{(\nu)}\right]$$

$$\le (\alpha_n + 2\beta_n)^2 p_n^{(\mu)} ,$$

and since $\beta_n = o(\alpha_n)$ and $\alpha_n = o(\sqrt{\hat{k}_n})$, which implies $\alpha_n^2 = o(n)$, it

follows from (3.2.11) that $B_3^{(n)} \to 0$. Similarly $B_4^{(n)} \to 0$ and $B_5^{(n)} \to 0$. Therefore $E\hat{X}^2_{n,2\ell_n} \to 0$, which along with (3.2.26) gives (3.2.14).

Finally, $|\hat{X}_{n,i}|$ is bounded by a numerical quantity K_n, where

$$K_n = O\left(\frac{\alpha_n}{\sqrt{\hat{k}_n}}\right),$$

which tends to zero, so that (3.2.15) holds. This completes the proof.

As an immediate corollary of this result we may in the obvious way obtain the following theorem giving the asymptotic joint distribution of several intermediate order statistics from the stationary sequence ξ_n.

<u>Theorem 3.2.2.</u> Let ξ_n be stationary with marginal d.f. F and associated independent sequence $\hat{\xi}_n$, and let $k_n^{(1)}, \ldots, k_n^{(m_1)}$, $k_n^{(m_1+1)}, \ldots, k_n^{(m)}$ be m intermediate rank sequences satisfying (3.2.1) and such that $\lambda_{\mu,\nu} = \lim_n \frac{k_n^{(\mu)}}{k_n^{(\nu)}}$ exists for $1 \le \mu \le \nu \le m_1$ and $\lambda_{\mu,\nu} = \lim_n \frac{k_n^{(\nu)}}{k_n^{(\mu)}}$ exists for $m_1+1 \le \mu \le \nu \le m$. Suppose there are constants $a_n^{(\mu)} > 0$, $b_n^{(\mu)}$ such that, as $n \to \infty$,

$$P\left(\frac{\hat{\xi}^{(n)}_{k_n^{(\mu)}} - b_n^{(\mu)}}{a_n^{(\mu)}} \le x\right) \to G^{(\mu)}(x) = \Phi\left(u^{(\mu)}(x)\right)$$

for all real x, for each $\mu = 1, \ldots, m_1$, and

$$P\left\{\frac{\hat{\xi}^{(n)}_{k_n^{(\mu)}} - b_n^{(\mu)}}{a_n^{(\mu)}} \le x\right\} \to G^{(\mu)}(x) = \Phi\left(u^{(\mu)}(x)\right)$$

for all real x, for each $\mu = m_1+1, \dots ,m$. Then if J holds for the set of sequences $\{u_n^{(1)}, \dots ,u_n^{(m)}\}$, where $u_n^{(\mu)} = a^{(\mu)}x_\mu + b_n^{(\mu)}$ for all real $x_1, \dots ,x_m$ for which each $u^{(\mu)}(x_\mu)$ is finite, or equivalently, for which $0 < G^{(\mu)}(x_\mu) < 1$, then

$$P\left\{\frac{{}_{k_n^{(\mu)}}\xi^{(n)} - b_n^{(\mu)}}{a_n^{(\mu)}} \le x_\mu \;,\quad \mu = 1, \dots ,m_1 \;;\right.$$

$$\left.\frac{\xi^{(n)}_{k_n^{(\mu)}} - b_n^{(\mu)}}{a_n^{(\mu)}} \le x_\mu \;,\quad \mu = m_1+1, \dots ,m\right\}$$

$$\to \Phi_{\Sigma_1}\left(u^{(1)}(x_1), \dots ,u^{(m_1)}(x_{m_1})\right)$$

$$\cdot \Phi_{\Sigma_2}\left(u^{(m_1+1)}(x_{m_1+1}), \dots ,u^{(m)}(x_m)\right)$$

as $n \to \infty$, where Φ_{Σ_1} and Φ_{Σ_2} are the zero-mean multinormal laws with respective covariance matrices given by (3.2.8) and (3.2.9). That is, the random vector $\left({}_{k_n^{(1)}}\xi^{(n)}, \dots , {}_{k_n^{(m_1)}}\xi^{(n)}, \xi^{(n)}_{k_n^{(m_1+1)}}, \dots ,\xi^{(n)}_{k_n^{(m)}}\right)$ has the same limiting distribution, and under the same normalization, as the random vector $\left({}_{k_n^{(1)}}\hat{\xi}^{(n)}, \dots , {}_{k_n^{(m_1)}}\hat{\xi}^{(n)}, \hat{\xi}^{(n)}_{k_n^{(m_1+1)}}, \dots ,\hat{\xi}^{(n)}_{k_n^{(m)}}\right)$.

(For each component of these random vectors, the appropriate normalization is that for the corresponding one-dimensional limiting distribution.)

For stationary normal sequences ξ_n we have found it possible to formulate conditions on the covariances $r_n = E\xi_1\xi_{n+1}$ sufficient for the assumption J to hold. This may be done by first establishing a more general form of Lemmas 2.4.1 and 2.4.2 involving different "u" levels, and working directly with the mixing condition (3.2.7), the m-dimensional analogue of (2.3.6) used in the one-dimensional case. This latter of course would lead to our again assuming the covariance sequence r_n tends to zero exponentially. However, the calculations are rather complicated and moreover, unfortunately involve making further restrictions upon the several intermediate rank sequences considered, as well as upon $\delta = \sup_{n\geq 1} |r_n|$. Because of this somewhat unappealing situation we will not present this development here.

CHAPTER IV

REPRESENTATIONS OF INTERMEDIATE ORDER STATISTICS

4.1. Introduction: The Bahadur Representation of Sample Quantiles.

For a fixed number λ in the unit interval $(0,1)$ let F be a distribution having a unique λ-quantile x_λ such that $F(x_\lambda) = \lambda$. In fact suppose that in some neighborhood $(x_\lambda-\delta, x_\lambda+\delta)$, F has a continuous derivative F' such that $F'(x_\lambda) > 0$. Then, as we have noted in Section 1.3, F belongs to the domain of attraction of the standard normal law for any sequence k_n of smallest ranks for which $\frac{k_n}{n} \to \lambda$, or equivalently, for any sequence k_n of largest ranks for which $\frac{n-k_n+1}{n} \to \lambda$.

Suppose that ξ_n is an i.i.d. sequence with marginal d.f. F. Let $F_n(x,\omega) = n^{-1} \sum_{i=1}^{n} I_{[\xi_i \le x]}(\omega)$ denote the empirical distribution of the sample $\xi_1, \ldots, \xi_n$. Under the additional assumptions that F has a bounded second derivative F'' in the neighborhood $(x_\lambda-\delta, x_\lambda+\delta)$ and that the central rank sequence k_n satisfies

$$(4.1.1) \qquad k_n = n\lambda + o(n^{\frac{1}{2}} \log n) \quad \text{as} \quad n \to \infty,$$

Bahadur (1966) proved the following representation of the k_n-th smallest order statistic:

$$\xi_{k_n}^{(n)}(\omega) = x_\lambda + \frac{k_n - nF_n(x_\lambda,\omega)}{nF'(x_\lambda)} + O(n^{-3/4} \log n)$$

with probability one (w.p.1). Then defining the sample λ-quantile Y_n of $\xi_1, \ldots, \xi_n$ as an average of the two order statistics $\xi^{(n)}_{k_n^{(1)}}$ and $\xi^{(n)}_{k_n^{(2)}}$, where the central rank sequences $k_n^{(1)} = [n\lambda]$ and $k_n^{(2)} = [n\lambda] + 1$ *each obviously satisfy* (4.1.1), he obtained

$$(4.1.2) \qquad Y_n(\omega) = x_\lambda + \frac{\lambda - F_n(x_\lambda, \omega)}{F'(x_\lambda)} + R_n(\omega)$$

where $R_n(\omega) = O(n^{-3/4} \log n)$ w.p.1. From the representation (4.1.2) it follows immediately by the ordinary central limit theorem for sums of i.i.d. variables having a finite second moment that the sample quantile Y_n is asymptotically normal with mean x_λ and variance $\frac{\lambda(1-\lambda)}{n[F'(x_\lambda)]^2}$. Moreover, it also follows that the law of the iterated logarithm holds for Y_n ; that is,

$$P\left\{\overline{\lim_n} \pm \frac{n^{\frac{1}{2}}(Y_n - x_\lambda)F'(x_\lambda)}{\left(2\lambda(1-\lambda)\log\log n\right)^{\frac{1}{2}}} = 1\right\} = 1.$$

The representation (4.1.2) has been extended by Sen (1968) to sample quantiles Y_n from m-dependent sequences not necessarily identically distributed, under certain conditions, and to those from stationary ϕ-mixing sequences (see Sen (1972)). From these representations Sen also gave further conditions which are sufficient for obtaining the asymptotic normality of Y_n as well as the corresponding iterated logarithm result. Additionally, Ghosh (1971) has found a representation of Y_n from independent or m-dependent sequences with a weaker convergence rate for the remainder term R_n , and in fact an "in probability" rather than an almost sure convergence. This latter is valid, however, under fewer assumptions, including requiring the existence of only the first derivative F' in a neighborhood of x_λ ,

and no longer than the sample quantile Y_n be defined in terms of central order statistics $\xi_{k_n}^{(n)}$ for which k_n satisfies (4.1.1). Moreover, Ghosh's procedure is substantially simpler than that of Bahadur and Sen (1968), and the representation with the weaker convergence rate is sufficient for many purposes, including obtaining the asymptotic normality of Y_n .

Using techniques similar to those of the above authors, we can obtain analagous representations for intermediate order statistics. For convenient comparisons of our methods and results with those for sample λ-quantiles, we deal exclusively with k_n-th smallest terms $\xi_{k_n}^{(n)}$, although we could just as well obtain corresponding results for largest terms. In establishing these results a crucial assumption we will be making in this chapter is that the marginal d.f. F of ξ_1 (and each marginal d.f. of a non-identically distributed sequence ξ_n) has a finite left endpoint x_0 , with additional assumptions to be made upon the behavior of $F(x)$ as x approaches x_0 , including those given in the hypothesis of Theorem 1.3.3. These of course will correspond in the intermediate case to the fact in the central case that λ-quantiles x_λ are necessarily finite, for $0 < \lambda < 1$, and to conditions upon the behavior of the d.f. F in a neighborhood of x_λ .

We begin with the development of a representation for intermediate order statistics $\xi_{k_n}^{(n)}$ from independent sequences in the next section, and in Section 4.3 we consider a weaker representation under fewer assumptions, which is valid also for the case of m-dependent and not necessarily identically distributed sequences. Also in this chapter we provide, as corollaries to our representation theorems, additional results dealing with the asymptotic behavior of intermediate order

statistics; specifically, we give a law of the iterated logarithm in the i.i.d. sequence case, and for m-dependent sequences we indicate the asymptotic distribution of $\xi_{k_n}^{(n)}$, which is already established for i.i.d. sequences (Theorem 1.3.3).

We conjecture that our results can be extended to intermediate order statistics from ϕ-mixing sequences, using methods similar to those of Sen (1972) in establishing an almost sure representation for sample λ-quantiles from such dependent sequences. However we will not consider this problem here.

4.2. I.i.d. Sequences.

Let F be a d.f. with a finite left endpoint x_0 such that $F(x_0) = 0$, and suppose that inside an interval $(x_0, x_0+\delta)$, F is twice differentiable with F'' bounded and such that $\lim_{x \downarrow x_0} F'(x)$ exists and is positive. Suppose that ξ_n is an i.i.d. sequence of r.v.'s with marginal d.f. F, and let k_n be an intermediate rank sequence subject to the restriction

$$(4.2.1) \qquad \frac{k_n}{\log n} \to \infty \quad \text{as} \quad n \to \infty.$$

Define a sequence of positive numbers

$$a_n = n^{-1} k_n^{\frac{1}{2}} (1+\varepsilon)^{\frac{1}{2}} \log^{\frac{1}{2}} n,$$

where $\varepsilon > 0$, and a sequence of integers b_n satisfying

$$b_n \sim \left(\frac{k_n}{\log n}\right)^{\frac{1}{4}} \quad \text{as} \quad n \to \infty.$$

Let x_n be such that $F(x_n) = \frac{k_n}{n}$. (x_n exists and is unique for

large n, by the assumed differentiability conditions on F.) Let

$$F_n(x,\omega) = n^{-1} \sum_{i=1}^{n} I_{[\xi_i \le x]}(\omega)$$

be the empirical d.f. of the sample $\xi_1, \ldots, \xi_n$, and write

$$G_n(x,\omega) = F_n(x,\omega) - F(x) - F_n(x_n,\omega) + \frac{k_n}{n}$$

and

$$H_n(\omega) = \sup_{x_n - a_n \le x \le x_n + a_n} |G_n(x,\omega)|.$$

Most of our development is then given in the following two lemmas.

Lemma 4.2.1. With probability one,

$$H_n(\omega) = O(n^{-1} k_n^{1/4} \log^{3/4} n) \quad \text{as} \quad n \to \infty.$$

Proof: Let N_1 be an integer such that x_n is uniquely defined for $n \ge N_1$. For $n \ge N_1$ and for any integer r, let $\eta_{r,n} = x_n + a_n b_n^{-1} r$. By the monotonicity of F and F_n we have for $x \in [\eta_{r,n}, \eta_{r+1,n}]$,

$$F_n(\eta_{r,n}, \omega) - F(\eta_{r+1,n}) \le F_n(x,\omega) - F(x) \le F_n(\eta_{r+1,n}, \omega) - F(\eta_{r,n}).$$

Thus

$$\begin{aligned} H_n(\omega) &\le \max_{-b_n \le r \le b_n} \{|F_n(\eta_{r,n}, \omega) - F(\eta_{r,n}) - F_n(x_n, \omega) + \frac{k_n}{n}|\} \\ &\quad + \max_{-b_n \le r \le b_n - 1} \{F(\eta_{r+1,n}) - F(\eta_{r,n})\} \end{aligned}$$

$$= K_n(\omega) + \alpha_n, \tag{4.2.2}$$

say. Since $\eta_{r+1,n} - \eta_{r,n} = a_n b_n^{-1}$, $\eta_{b_n,n} = x_n + a_n \to x_0$, and by the assumed conditions upon F in the fixed interval $(x_0, x_0+\delta)$, we have that

(4.2.3) $$\alpha_n = O(n^{-1} k_n^{1/4} \log^{3/4} n).$$

We will use the following inequality of Bernstein (see Bahadur (1966)): Let $B(n,p)$ be a binomial r.v. with parameters n and p, for some integer $n \geq 1$ and for $0 \leq p \leq 1$. Then for every $t > 0$,

(4.2.4) $$P\{|B(n,p) - np| \geq t\} \leq 2 \exp(-h),$$

where

$$h = h(n,p,t) = \frac{t^2}{2\left\{np(1-p) + \frac{t}{3}\max\{p, 1-p\}\right\}}.$$

Let $C_1 > \lim_{x \downarrow x_0} F'(x) > 0$. We may choose $N_2 \geq N_1$ such that $F(x_n + a_n) - F(x_n) < C_1 a_n$ and $F(x_n) - F(x_n - a_n) < C_1 a_n$, for all $n \geq N_2$. Now for any $n \geq N_1$ and any integer r,

$$G_n(\eta_{r,n}) = \begin{cases} n^{-1} \sum_{i=1}^{n} I_{[x_n < \xi_i \leq \eta_{r,n}]} - p_{r,n}, & \text{if} \quad r \geq 0 \\ -n^{-1} \sum_{i=1}^{n} I_{[\eta_{r,n} < \xi_i \leq x_n]} + p_{r,n}, & \text{if} \quad r \leq -1, \end{cases}$$

where

$$p_{r,n} = |F(\eta_{r,n}) - \frac{k_n}{n}|.$$

Letting $t_n = C_2 n^{-1} k_n^{1/4} \log^{3/4} n$ for some $C_2 > 0$, we therefore have by (4.2.4),

$$P\{|G_n(\eta_{r,n})| \geq t_n\} \leq 2 \exp(-h_{r,n}),$$

where

$$h_{r,n} = \frac{n^2 t_n^2}{2\left\{np_{r,n}(1-p_{r,n}) + \frac{nt_n}{3}\max\{p_{r,n}, 1-p_{r,n}\}\right\}}$$

$$\geq \frac{nt_n^2}{2p_{r,n} + t_n} .$$

Since $n \geq N_2$ and $|r| \leq b_n$ imply $p_{r,n} < C_1 a_n$, we have

$$P(|G_n(\eta_{r,n})| \geq t_n) \leq 2 \exp(-\delta_n)$$

for $n \geq N_2$ and $|r| \leq b_n$, where

$$\delta_n = \frac{nt_n^2}{2C_1 a_n + t_n} ,$$

which does not depend on r. Therefore

$$P(K_n \geq t_n) \leq \sum_{r=-b_n}^{b_n} P(|G_n(\eta_{r,n})| \geq t_n)$$

$$\leq 4\, b_n \exp(-\delta_n)$$

$$< 4\, n \exp(-\delta_n).$$

An easy calculation shows that $\delta_n \geq \dfrac{C_2^2}{2C_1(1+\varepsilon)^{\frac{1}{2}}+1} \log n$ for $n \geq N_3$, say, so that for given C_1 and ε, choosing C_2 sufficiently large such that $\dfrac{C_2^2}{2C_1(1+\varepsilon)^{\frac{1}{2}}+1} > 2$, we obtain

$$\sum_{n \geq N_1} P(K_n \geq t_n) < \infty.$$

Thus by the Borel-Cantelli lemma,

$$P(K_n \geq t_n \text{ infinitely often}) = 0,$$

and since $t_n = C_2\, n^{-1}\, k_n^{\frac{1}{4}} \log^{3/4} n$, the statement of the lemma follows from (4.2.2) and (4.2.3).

Lemma 4.2.2. With probability one,

$$x_n - a_n \le \xi_{k_n}^{(n)}(\omega) \le x_n + a_n$$

for all large n (that is, for all $n \ge$ some $N(\omega)$). Note that $x_0 < x_n - a_n$ for all large n, by the choice of a_n, the assumed conditions on F, and (4.2.1).

Proof: First we have

$$\begin{aligned} P(\xi_{k_n}^{(n)} < x_n - a_n) &\le P\Big(\sum_{i=1}^{n} I_{[\xi_i \le x_n - a_n]} \ge k_n\Big) \\ &= P\Big(\sum_{i=1}^{n} I_{[\xi_i \le x_n - a_n]} - np_n \ge k_n - np_n\Big) \\ &\le P\Big(\sum_{i=1}^{n} I_{[\xi_i \le x_n - a_n]} - np_n > C_3\, na_n\Big) \end{aligned}$$

where $p_n = F(x_n - a_n)$, for some $C_3 > 0$, and for all large n, since $a_n = o(\frac{k_n}{n})$ by (4.2.1), and since F' exists and is bounded away from zero in some fixed interval. Then by (4.2.4),

$$\begin{aligned} P(\xi_{k_n}^{(n)} < x_n - a_n) &\le 2\exp\left(-\frac{C_3^2 n^2 a_n^2}{2np_n + C_3 na_n}\right) \\ &\le 2\exp\left(-\frac{C_4 n^2 a_n^2}{k_n}\right) \end{aligned}$$

for some $C_4 > 0$ (not depending on ε) and for large n, since $p_n \le \frac{k_n}{n}$ and $a_n = o(\frac{k_n}{n})$. But $k_n^{-1} n^2 a_n^2 = (1+\varepsilon)\log n$, so that if $\varepsilon > 0$ is chosen sufficiently large, we obtain

(4.2.5) $$\sum_{n \ge N_1} P(\xi_{k_n}^{(n)} < x_n - a_n) < \infty.$$

In a similar way we have

$$P(\xi_{k_n}^{(n)} > x_n + a_n) \le P\Big(\sum_{i=1}^{n} I_{[\xi_i \le x_n + a_n]} < k_n\Big)$$

$$= P\left(\sum_{i=1}^{n} I_{[\xi_i > x_n + a_n]} - np_n > -k_n + n(1-p_n)\right)$$

$$\le P\left(\sum_{i=1}^{n} I_{[\xi_i > x_n + a_n]} - np_n > C_3\, na_n\right)$$

for large n, where now $p_n = 1 - F(x_n + a_n)$. Then the relations $F(x_n + a_n) < \frac{k_n}{n} + C_1 a_n$ and $a_n = o\left(\frac{k_n}{n}\right)$ along with (4.2.4) lead to

$$P\left(\xi_{k_n}^{(n)} > x_n + a_n\right) \le 2 \exp\left(- \frac{C_5 n^2 a_n^2}{k_n}\right)$$

for some $C_5 > 0$ and for large n, so if $\varepsilon > 0$ is sufficiently large,

(4.2.6) $$\sum_{n \ge N_1} P\left(\xi_{k_n}^{(n)} > x_n + a_n\right) < \infty.$$

Combining (4.2.5) and (4.2.6) and applying the Borel-Cantelli lemma completes the proof.

Our main result is now easily obtained.

Theorem 4.2.3. Let ξ_n be an i.i.d. sequence of r.v.'s with marginal d.f. F having a finite left endpoint x_0 such that $F(x_0) = 0$, and such that in an interval $(x_0, x_0+\delta)$, F has a bounded second derivative, and $\lim_{x \downarrow x_0} F'(x) > 0$. Let k_n be an intermediate rank sequence such that $\frac{k_n}{\log n} \to \infty$, and for sufficiently large n define x_n by $F(x_n) = \frac{k_n}{n}$. Then

(4.2.7) $$\xi_{k_n}^{(n)}(\omega) = x_n + \frac{k_n - nF_n(x_n, \omega)}{nF'(x_n)} + R_n(\omega),$$

where

(4.2.8) $R_n(\omega) = O(n^{-1}\, k_n^{1/4} \log^{3/4} n)$ with probability one.

Proof: From Lemmas 4.2.1 and 4.2.2 it follows that w.p.1,

$$|F_n(\xi^{(n)}_{k_n}(\omega),\ \omega) - F(\xi^{(n)}_{k_n}(\omega)) - F_n(x_n,\omega) + \frac{k_n}{n}| = O(n^{-1}\ k_n^{1/4}\ \log^{3/4}\ n).$$

Since F has a bounded second derivative in $(x_0,\ x_0+\delta)$ and since $a_n = o(1)$, Lemma 4.2.2 gives

$$F(\xi^{(n)}_{k_n}(\omega)) = \frac{k_n}{n} + (\xi^{(n)}_{k_n}(\omega) - x_n)F'(x_n) + O(n^{-2}\ k_n\ \log\ n)$$

w.p.1. Then since we may assume (w.p.1) that $F_n(\xi^{(n)}_{k_n}(\omega),\ \omega) = \frac{k_n}{n}$, noting that $F(x_0) = 0$, and from the assumption that $F'(x_n)$ is bounded away from zero for large n, the theorem follows.

If k_n satisfies $\frac{k_n}{\log^3\ n} \to \infty$, which is slightly more restrictive than (4.2.1), then as is evident the representation (4.2.7)-(4.2.8) verifies the asymptotic normality of $\xi^{(n)}_{k_n}$ stated in Theorem 1.3.3, under stronger conditions upon the marginal d.f. F. However we may use our representation to establish as a new result a law of the iterated logarithm for $\xi^{(n)}_{k_n}$. This has been obtained explicitly for intermediate order statistics from i.i.d. sequences of r.v.'s uniformly distributed on the unit interval by Kiefer (1972, Theorem 5), under the restriction that $\frac{k_n}{\log\ \log\ n} \to \infty$. Translating his result according to the familiar relation

$$F_n(x_n,\omega) = n^{-1} \sum_{i=1}^{n} I_{[F(\xi_i) \leq \frac{k_n}{n}]}(\omega)\ ,$$

noting that $P(F(\xi_i) \leq \frac{k_n}{n}) = \frac{k_n}{n}$ for all sufficiently large n, and using the representation (4.2.7)-(4.2.8), valid under our assumptions upon the marginal d.f. F, we easily obtain the following theorem.

Theorem 4.2.4. Let ξ_n be an i.i.d. sequence with marginal d.f. F satisfying the conditions of Theorem 4.2.3. If k_n satisfies the restriction

$$\frac{k_n(\log\log n)^2}{\log^3 n} \to \infty \quad \text{as} \quad n \to \infty,$$

then

$$P\left(\overline{\lim_n} \pm \frac{nF'(x_n)\left(\xi^{(n)}_{k_n} - x_n\right)}{(2k_n \log\log n)^{\frac{1}{2}}} = 1\right) = 1.$$

4.3. Independent or m-dependent Sequences Under Weaker Conditions.

Following a procedure used by Ghosh (1971) to extend the Bahadur representation of sample λ-quantiles from independent sequences under weaker conditions, we will derive in this section a similar result for intermediate order statistics, which will be valid also in the case of m-dependent sequences. Specifically, the assumptions which will no longer be required include the existence and boundedness of the second derivative of the marginal d.f. of ξ_1 in the neighborhood $(x_0, x_0+\delta)$ adjacent to its left endpoint, as well as the restriction (4.2.1) upon how slowly the intermediate rank sequence k_n tends to infinity. By removing these conditions we will be left with only those actually given in the hypothesis of Theorem 1.3.3, which established the asymptotic normality of $\xi^{(n)}_{k_n}$ when the ξ_n are independent. The resulting representation however will correspondingly be for the most part weaker, in that a slower convergence rate for the remainder term R_n, and moreover, an "in probability" rather than an almost sure convergence statement, will be obtained. But the rate will still be sufficiently

fast to insure the asymptotic normality of $\xi_{k_n}^{(n)}$ in the i.i.d. case as an immediate corollary.

The method we use is somewhat simpler than that followed in Section 4.2, and in the present section it is of only slight inconvenience to in fact consider a more general setting in which the identically distributed assumption is relaxed to some extent. We therefore do this, and following our development we indicate the specialization to i.i.d. as well as to m-dependent stationary sequences.

Thus let ξ_n be an m-dependent sequence $(m \geq 0)$ with marginal d.f.'s $F_n(x) = P(\xi_n \leq x)$. Suppose for each $n \geq 1$, x_0 is the finite left endpoint of F_n such that $F_n(x_0) = 0$, and in an interval $(x_0, x_0+\delta)$, F_n has a continuous derivative F_n' such that $\lim_{x \downarrow x_0} F_n'(x)$ exists and is positive. Now define

$$\bar{F}_{(n)}(x) = n^{-1} \sum_{i=1}^{n} F_i(x).$$

Given an intermediate rank sequence k_n , for each $i = 1, \ldots ,n$ let $x_{n,i}$ be such that

$$F_i(x_{n,i}) = \frac{k_n}{n} ,$$

and let $\tilde{x}_n$ be such that

$$\bar{F}_{(n)}(\tilde{x}_n) = \frac{k_n}{n} .$$

(These exist uniquely for all sufficiently large n .) Further assume that

$$(4.3.1) \quad x_{n,i} - \tilde{x}_n = o\left(\frac{k_n^{1/2}}{n}\right), \text{ uniformly in } i = 1, \ldots ,n,$$

as $n \to \infty$ (which implies that $\tilde{x}_n \to x_0$), and that

$$\lim_n n^{-1} \sum_{i=1}^{n} F_i'(\tilde{x}_n) = \lim_n \overline{F}'_{(n)}(\tilde{x}_n) > 0. \tag{4.3.2}$$

(The limit is assumed to exist.) Finally, assume that

$$\overline{\lim_n}\ \frac{k_n}{\sigma_n^2} \to \infty, \tag{4.3.3}$$

where

$$\sigma_n^2 = \mathrm{Var}\Big(\sum_{i=1}^{n} I_{[\xi_i \le x_{n,i}]}\Big).$$

<u>Theorem 4.3.1.</u> Under the above conditions,

$$\xi_{k_n}^{(n)} = \tilde{x}_n + \frac{k_n - \sum_{i=1}^{n} I_{[\xi_i \le x_{n,i}]}}{n\overline{F}'_{(n)}(\tilde{x}_n)} + R_n\ , \tag{4.3.4}$$

where $\frac{n}{k_n^{1/2}} R_n \xrightarrow{p} 0.$

<u>Proof</u>: We first assume that the ξ_n are independent and later indicate the modifications to be made for the more general m-dependent case.

For any real t we have for sufficiently large n,

$$P\left(\frac{n}{k_n^{1/2}}(\xi_{k_n}^{(n)} - \tilde{x}_n) \le t\right) = P\left(\sum_{i=1}^{n} I_{\left[\xi_i \le \tilde{x}_n + \frac{tk_n^{1/2}}{n}\right]} \ge k_n\right)$$

$$= P\left(\frac{1}{k_n^{1/2}\overline{F}'_{(n)}(\tilde{x}_n)} \sum_{i=1}^{n} \left\{F_i\left(\tilde{x}_n + \frac{tk_n^{1/2}}{n}\right) - I_{\left[\xi_i \le \tilde{x}_n + \frac{tk_n^{1/2}}{n}\right]}\right\}\right.$$

$$\left.\le \frac{1}{k_n^{1/2}\overline{F}'_{(n)}(\tilde{x}_n)} \sum_{i=1}^{n} \left\{F_i\left(\tilde{x}_n + \frac{tk_n^{1/2}}{n}\right) - \frac{k_n}{n}\right\}\right)$$

$$(4.3.5) \qquad = P(Z_{t,n} \le t_n),$$

say. Since $\overline{F}_{(n)}\left(\tilde{x}_n + \frac{tk_n^{\frac{1}{2}}}{n}\right) = \frac{k_n}{n} + \frac{tk_n^{\frac{1}{2}}}{n}\overline{F}'_{(n)}(\tilde{x}_n) + o\left(\frac{k_n^{\frac{1}{2}}}{n}\right)$, we have that

$$t_n = \frac{n}{k_n^{\frac{1}{2}}\overline{F}'_{(n)}(\tilde{x}_n)}\left\{\frac{tk_n^{\frac{1}{2}}}{n}\overline{F}'_{(n)}(\tilde{x}_n) + o\left(\frac{k_n^{\frac{1}{2}}}{n}\right)\right\}$$

$$(4.3.6) \qquad \to t, \quad \text{as} \quad n \to \infty.$$

Suppose $t > 0$. Let

$$W_n = \frac{1}{k_n^{\frac{1}{2}}\overline{F}'_{(n)}(\tilde{x}_n)}\sum_{i=1}^{n}\left\{\frac{k_n}{n} - I_{[\xi_i \le x_{n,i}]}\right\}.$$

By (4.3.1) we have for large n that $x_{n,i} < \tilde{x}_n + \frac{tk_n^{\frac{1}{2}}}{n}$, so that

$$(4.3.7) \quad W_n - Z_{t,n} = \frac{1}{k_n^{\frac{1}{2}}\overline{F}'_{(n)}(\tilde{x}_n)}\sum_{i=1}^{n}\left\{I_{\left[x_{n,i} < \xi_i \le \tilde{x}_n + \frac{tk_n^{\frac{1}{2}}}{n}\right]} - p_{n,i}\right\},$$

where

$$p_{n,i} = F_i\left(\tilde{x}_n + \frac{tk_n^{\frac{1}{2}}}{n}\right) - \frac{k_n}{n}.$$

Since

$$E(W_n - Z_{t,n})^2 = \frac{1}{k_n[\overline{F}'_{(n)}(\tilde{x}_n)]^2}\sum_{i=1}^{n} p_{n,i}(1 - p_{n,i})$$

$$\le \frac{1}{k_n[\overline{F}'_{(n)}(\tilde{x}_n)]^2}\sum_{i=1}^{n} p_{n,i},$$

and since

$$\sum_{i=1}^{n} p_{n,i} = n\,\overline{F}_{(n)}\left(\tilde{x}_n + \frac{tk_n^{\frac{1}{2}}}{n}\right) - k_n$$

$$= k_n + tk_n^{\frac{1}{2}}\overline{F}'_{(n)}(\tilde{x}_n) + o(k_n^{\frac{1}{2}}) - k_n,$$

it follows that $E(W_n - Z_{t,n})^2 \to 0$ as $n \to \infty$, so that

$$(4.3.8) \qquad W_n - Z_{t,n} \xrightarrow{p} 0.$$

Similarly, (4.3.8) holds for $t \le 0$. Also, by the usual central limit theorem for row sums from triangular arrays, it is easily seen that

$$(4.3.9) \qquad W_n \Rightarrow N\left(0,\ [\lim_n \overline{F}'_{(n)}(\tilde{x}_n)]^{-2}\right),$$

so that in particular the sequence of distributions $\{\mathcal{L}(W_n),\ n\ge 1\}$ is tight.

Let

$$V_n = \frac{n}{k_n^{\frac{1}{2}}}\left(\xi_{k_n}^{(n)} - \tilde{x}_n\right).$$

Then by (4.3.5)-(4.3.9) we have that V_n and W_n satisfy the conditions of Lemma 1 of Ghosh (1971), that is, the collection $\{\mathcal{L}(W_n)\}$ is tight, and for all real t and all $\varepsilon > 0$,

$$\lim_n P(V_n \le t,\ W_n \ge t+\varepsilon) = 0$$

and

$$\lim_n P(V_n \ge t+\varepsilon,\ W_n \le t) = 0 .$$

Therefore, $V_n - W_n \xrightarrow{p} 0$. Hence for R_n defined by (4.3.4),

$$\begin{aligned}
\frac{n}{k_n^{\frac{1}{2}}} R_n &= \frac{n}{k_n^{\frac{1}{2}}}\left\{\xi_{k_n}^{(n)} - \tilde{x}_n - \left(\frac{k_n - \sum_{i=1}^{n} I_{[\xi_i \le x_{n,i}]}}{n\overline{F}'_{(n)}(\tilde{x}_n)}\right)\right\} \\
&= \frac{n}{k_n^{\frac{1}{2}}}\left(\xi_{k_n}^{(n)} - \tilde{x}_n\right) - \frac{1}{k_n^{\frac{1}{2}}\overline{F}'_{(n)}(\tilde{x}_n)} \sum_{i=1}^{n}\left\{\frac{k_n}{n} - I_{[\xi_i \le x_{n,i}]}\right\} \\
&= V_n - W_n \\
&\xrightarrow{p} 0 .
\end{aligned}$$

We now indicate the modifications to be made in order to establish the theorem for m-dependent sequences. We need to show

(4.3.10) W_n satisfies the first condition of Lemma 1 of Ghosh (1971); that is, the sequence $\{\mathcal{L}(W_n)\}$ is tight,

and

$$E(W_n - Z_{t,n})^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{4.3.11}$$

To show (4.3.11) we use the following result of Rosen (1967): If $Y_1, \ldots, Y_n$ are m-dependent $(n \geq m)$ and have finite variances, then

$$\mathrm{Var}\Big(\sum_{i=1}^{n} Y_i\Big) \leq (2m+1) \sum_{i=1}^{n} \mathrm{Var}\, Y_i\ . \tag{4.3.12}$$

Applying this to (4.3.7) yields

$$E(W_n - Z_{t,n})^2 \leq \frac{1}{k_n\{\overline{F}'_{(n)}(\tilde{x}_n)\}^2}(2m+1) \sum_{i=1}^{n} p_{n,i}\ ,$$

which tends to zero as in the independent case.

In order to show (4.3.10) we employ the following central limit theorem for row sums from triangular arrays of r.v.'s with m-dependent rows ($m \geq 0$ fixed), due to Rosen (1967): Let $\{X_{n,i}\}_{n\geq 1, 1\leq i\leq n}$, with d.f.'s $F_{n,i}(x) = P(X_{n,i} \leq x)$ be an array of r.v.'s with $X_{n,1}, \ldots, X_{n,n}$ m-dependent for each $n \geq m+1$, and let $S_n = \sum_{i=1}^{n} X_{n,i}$. If

$$E\, X_{n,i} = 0 \quad \text{for all } n,i\ ,$$

$$\mathrm{Var}\, S_n = 0 \quad \text{for all } n\ ,$$

$$\lim_n \sum_{i=1}^{n} \int_{|x|>\varepsilon} x^2\, dF_{n,i}(x) = 0 \quad \text{for all } \varepsilon > 0,$$

and

$$\overline{\lim_n} \sum_{i=1}^{n} \text{Var } X_{n,i} < \infty ,$$

then

$$S_n \Rightarrow N(0,1) .$$

Now, we first note that

$$(4.3.13) \quad W_n = - \frac{1}{k_n^{\frac{1}{2}} F'_{(n)}(\tilde{x}_n)} \sigma_n \sum_{i=1}^{n} \left\{ \frac{I_{[\xi_i \le x_{n,i}]} - \frac{k_n}{n}}{\sigma_n} \right\} .$$

If (4.3.2) holds, then it is easily shown that the array $\{X_{n,i}\}$, for n large and $1 \le i \le n$, given by

$$X_{n,i} = \frac{I_{[\xi_i \le x_{n,i}]} - \frac{k_n}{n}}{\sigma_n} ,$$

satisfies the assumptions of the above central limit theorem, so that

$$S_n = \sum_{i=1}^{n} X_{n,i} \Rightarrow N(0,1) .$$

Then using (4.3.2) again we have

$$\frac{\sigma_n^2}{k_n} \le \frac{1}{k_n} (2m+1) \frac{nk_n}{n}\left(1 - \frac{k_n}{n}\right) = (2m+1)\big(1 + o(1)\big) ,$$

and it is clear from (4.3.13) that (4.3.10) holds; that is, the collection $\{\mathfrak{L}(W_n)\}$ is tight (but not necessarily weakly convergent). The proof of the theorem is now complete.

For identically distributed sequences ξ_n we may replace the $x_{n,i}$, $i = 1, \ldots ,n$, by $\tilde{x}_n = x_n$, say, and thus disregard (4.3.1); additionally in this instance the assumption (4.3.2) is redundant. For

i.i.d. sequences (4.3.3) is trivially satisfied and hence we have a representation generally weaker than that in Theorem 4.2.3, but using fewer assumptions on the marginal d.f. and having no restriction upon the intermediate rank sequence k_n. Also, for stationary m-dependent sequences $(m \geq 1)$ (4.3.3) becomes

$$\overline{\lim_n}\ k_n\left\{k_n + 2\sum_{i=1}^{m}(n-i)\left[P(\xi_1 \leq x_n,\ \xi_{1+i} \leq x_n) - \left(\frac{k_n}{n}\right)^2\right]\right\}^{-1} < \infty ,$$

which holds if

$$\frac{2}{k_n}\sum_{i=1}^{m}(n-i)\left[P(\xi_1 \leq x_n,\ \xi_{1+i} \leq x_n) - \left(\frac{k_n}{n}\right)^2\right]$$

is bounded away from -1 for all large n.

Finally we note that the asymptotic normality of $\xi_{k_n}^{(n)}$ easily follows as a corollary to the representation (4.3.4), by (4.3.3) and again upon using the central limit theorem of Rosen (1967). We state this formally as a theorem.

Theorem 4.3.2. Under the same conditions as Theorem 4.3.1, and with the previous notation,

$$\frac{n\bar{F}'_{(n)}(\tilde{x}_n)}{\sigma_n}\left(\xi_{k_n}^{(n)} - \tilde{x}_n\right) \Rightarrow N(0,1) .$$

CHAPTER V

FURTHER RESULTS

Two miscellaneous results relevant to our study of intermediate order statistics will be obtained in this chapter. These results may be regarded as supplementary to our previous work, in that we will not attempt to develop them to the fullest potential generality and usefulness. Instead, we let them serve to indicate directions for further theoretical investigations.

5.1. The Second Moment of the Remainder Term In the Intermediate Order Statistic Representation.

In this section we derive an asymptotic approximation to the second moment of the remainder term R_n appearing in the representation of the intermediate order statistic $\xi_{k_n}^{(n)}$ given in Theorem 4.2.3. Such an asymptotic expression has been obtained for the second moment of the remainder term appearing in the Bahadur representation (4.1.2) of sample λ-quantiles by Duttweiler (1973), who has discussed the particular application to a problem involving stochastic processes of interest in engineering, which motivated the consideration of the approximation.

In our situation we retain all of the assumptions upon the marginal d.f. F of ξ_1 made for the result (4.2.7)-(4.2.8), and

additionally, as did Duttweiler, we impose the requirement that $E\,\xi_1^2 = \int x^2\, dF(x) < \infty$. We also make an assumption slightly stronger than (4.2.1) as to how rapidly the intermediate rank sequence k_n tends to infinity; here we suppose that

$$\frac{k_n}{n^\theta} \to \infty$$

for some $\theta > 0$. Under these conditions we have found that Duttweiler's procedure can essentially be carried through for the intermediate ranks case, leading to an estimate of the quantity $E R_n^2$, which is nearly as precise as the corresponding estimate in the central ranks case. Whether our approximation has a similar useful application remains to be seen.

Owing to the basic fact that the expected value of the k-th smallest order statistic of a sample of n variables distributed uniformly on the unit interval $(0,1)$ is $\frac{k}{n+1}$ rather than $\frac{k}{n}$, it turns out that a substantial simplification in the procedure is effected by dealing with a representation for $\xi_{k_n}^{(n)}$ slightly different from that given in Theorem 4.2.3. In fact, let us consider, in place of (4.2.7), the representation

$$(5.1.1)\quad \xi_{k_n}^{(n)}(\omega) = x_n' + \frac{\frac{nk_n}{n+1} - \sum_{i=1}^{n} I_{[\xi_i \le x_n']}(\omega)}{nF'(x_n')} + R_n'(\omega),$$

where x_n' is defined by $F(x_n') = \frac{k_n}{n+1}$, for large n, F being the marginal d.f. of the i.i.d. sequence ξ_n. An inspection of the proofs in Section 4.2 reveals that, as should be intuitively obvious, x_n can be replaced by x_n' throughout, leading to the conclusion that for R_n' defined by (5.1.1), we have

$$R_n'(\omega) = O(n^{-1}\, k_n^{1/4} \log^{3/4} n) \text{ with probability one.}$$

The following preliminary result, which we state as a lemma, gives an exact expression for the second moment of R_n' when F is the uniform $(0,1)$ distribution. The details of its proof may be found in Duttweiler (1973).

Lemma 5.1.1. Let $U_k^{(n)}$ be the k-th smallest order statistic for the set $U_1,\ldots,U_n$ of n independent r.v.'s each distributed uniformly on $(0,1)$. Define

$$(5.1.2) \qquad \hat{U}_k^{(n)} = \frac{k}{n+1} + \frac{\sum_{i=1}^{n} I_{[U_i \geq \frac{k}{n+1}]} - n\left(1 - \frac{k}{n+1}\right)}{n}$$

and let

$$R = U_k^{(n)} - \hat{U}_k^{(n)} .$$

Then

$$ER^2 = \frac{2k}{n(n+1)}\left\{I_{\frac{k}{n+1}}(k,\ n+1-k) - I_{\frac{k}{n+1}}(k+1,\ n+1-k)\right\} - \frac{2k\left(1-\frac{k}{n+1}\right)}{n(n+1)(n+2)}$$

$$\leq \frac{2}{n}\left[\frac{k\left(1-\frac{k}{n+1}\right)}{(n+1)(n+2)}\right]^{\frac{1}{2}},$$

where $I_x(a,b)$ is the incomplete Beta function, defined by

$$I_x(a,b) = \frac{(a+b-1)!}{(a-1)!\,(b-1)!}\int_0^x y^{a-1}(1-y)^{b-1}dy.$$

Using this result we may now derive an approximation to the second moment of R_n' under the assumptions of Section 4.2. As noted earlier we will be closely following Duttweiler's proof for the sample λ-quantile case; however for clarity we will here present most of the details.

Theorem 5.2.2. Let F be a d.f. with finite left endpoint x_0 such that $F(x_0) = 0$, and suppose that inside an interval $(x_0,\ x_0+\delta)$, F

is twice differentiable with F'' bounded and such that $\lim_{x \downarrow x_0} F'(x)$ exists (finite) and is positive. Suppose ξ_n is an i.i.d. sequence with marginal d.f. F, and assume that $E\xi_1^2 = \int x^2 dF(x) < \infty$. Let k_n be an intermediate rank requence satisfying $\frac{k_n}{n^\theta} \to \infty$ for some $\theta > 0$, and let x_n' be such that $F(x_n') = \frac{k}{n+1}$ (for large n). Then for R_n' defined by (5.1.1), we have

$$E(R_n')^2 \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{k_n^{\frac{1}{2}}}{(nF'(x_n'))^2}, \quad \text{as } n \to \infty.$$

Proof: Let $U_1, U_2, \ldots$ be independent r.v.'s uniformly distributed on $(0,1)$ and let $U_{k_n}^{(n)}$ be the k_n-th smallest of $U_1, \ldots, U_n$. Define the quantile function $Q(\cdot)$ by

$$Q(u) = \sup\{x: F(x) \leq u\}.$$

Then for each $i \geq 1$, $Q(U_i)$ has d.f. F, and moreover, as in Duttweiler (1973) we have that

$$R_n' \approx Q\left(U_{k_n}^{(n)}\right) - x_n' - \frac{\frac{nk_n}{n+1} - \sum_{i=1}^{n} I_{[Q(U_i) \leq x_n']}}{nF'(x_n')},$$

where $\approx$ indicates having identical distributions.

From the assumptions on the d.f. F we may as well suppose that $F'(x)$ is positive in the interval $A = (x_0, x_0+\delta)$. Let $B = \{u: u=F(x), x \in A\}$. Then Q restricted to B is the inverse of F restricted to A. Since $x_n' \to x_0$, there is an integer N_1 such that $\frac{k_n}{n+1} \in B$ if $n \geq N_1$. Thus for $n \geq N_1$,

$$Q\left(\frac{k_n}{n+1}\right) = x_n', \tag{5.1.3}$$

(5.1.4) $\quad Q'\left(\frac{k_n}{n+1}\right)$ exists and equals $\left(F'(x_n')\right)^{-1}$,

and

(5.1.5) $\quad Q(u) \le x_n'$ if and only if $u \le \frac{k_n}{n+1}$.

Also, it is clear that Q'' exists and is bounded in B (see Van Zwet (1964, p. 24)). Moreover, we have the relation

$$\int_0^1 Q^2(u)\,du = E\xi_1^2 < \infty.$$

(To see this, first regard Q as a (Borel-) measurable transformation from the unit interval into the real line, and let μ be Lebesgue measure. Then

$$\int_0^1 Q^2(u)\,du = \int_{-\infty}^{\infty} x^2\, d(\mu Q^{-1})(x),$$

where the measure μQ^{-1} is defined by $\mu Q^{-1}(E) = \mu\left(Q^{-1}(E)\right)$ for Borel sets E. By merely considering bounded semiclosed intervals we may verify that μQ^{-1} is in fact the Lebesgue-Stieltjes measure corresponding to F .)

Now, for $n \ge 1$ and $u \in [0,1]$, define

$$H_n(u) = Q(u) - Q\left(\frac{k_n}{n+1}\right) - Q'\left(\frac{k_n}{n+1}\right)\left(u - \frac{k_n}{n+1}\right).$$

Then by (5.1.3)-(5.1.5), for $n \ge N_1$ we have that

$$R_n' \approx Q\left(\frac{k_n}{n+1}\right) + Q'\left(\frac{k_n}{n+1}\right)\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right) + H_n\left(U_{k_n}^{(n)}\right)$$

$$- x_n' - \frac{\frac{k_n}{n+1} - \sum_{i=1}^{n} I_{\left[U_i \le \frac{k_n}{n+1}\right]}}{nF'(x_n')}$$

$$= H_n\left(U_{k_n}^{(n)}\right) + \left(F'(x_n')\right)^{-1}\left\{U_{k_n}^{(n)} - \frac{k_n}{n+1} - \frac{\frac{nk_n}{n+1} - \sum_{i=1}^{n} I_{\left[U_i \le \frac{k_n}{n+1}\right]}}{n}\right\}$$

$$\approx H_n\left(U_{k_n}^{(n)}\right) + \left(F'(x_n')\right)^{-1}\left\{U_{k_n}^{(n)} - \frac{k_n}{n+1} - \frac{\sum_{i=1}^{n} I_{\left[U_i \ge \frac{k_n}{n+1}\right]} - n\left(1 - \frac{k_n}{n+1}\right)}{n}\right\}$$

$$(5.1.6) \quad = H_n\left(U_{k_n}^{(n)}\right) + \left(F'(x_n')\right)^{-1}\{U_{k_n}^{(n)} - \hat{U}_{k_n}^{(n)}\},$$

where $\hat{U}_{k_n}^{(n)}$ is given by (5.1.2). Now by Lemma 5.1.1,

$$E\left(U_{k_n}^{(n)} - \hat{U}_{k_n}^{(n)}\right)^2 = \frac{2k_n}{n(n+1)}\left(I_{\frac{k_n}{n+1}}(k_n, n+1-k_n) - I_{\frac{k_n}{n+1}}(k_n+1, n+1-k_n)\right) + O\left(\frac{k_n}{n^3}\right).$$

From Abramowitz and Stegun (1964, Equations 26.5.16 and 6.1.37) and Wilks (1962, p. 177) we have the relation

$$I_x(a,b) - I_x(a+1, b) = \frac{(a+b-1)!}{a!(b-1)!} x^a (1-x)^b$$

for positive integers a and b, and Stirling's formula

$$n! = e^{-n} n^{n+\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left(1 + O(n^{-1})\right).$$

Therefore

$$E\left(U_{k_n}^{(n)} - \hat{U}_{k_n}^{(n)}\right)^2 = \frac{2k_n}{n(n+1)} \cdot \frac{n!}{k_n!(n-k_n)!} \cdot \left(\frac{k_n}{n+1}\right)^{k_n}\left(\frac{n+1-k_n}{n+1}\right)^{n+1-k_n}$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cdot \frac{k_n^{\frac{1}{2}}}{n(n+1)} \cdot \left(\frac{n-k_n}{n}\right)^{\frac{1}{2}}\left(\frac{n}{n+1}\right)^{n+1}\left(\frac{n+1-k_n}{n-k_n}\right)^{n+1-k_n}$$

$$\cdot \left(1 + O\left(\frac{1}{n}\right)\right) + O\left(\frac{k_n}{n^3}\right)$$

$$(5.1.7) \qquad = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{k_n^{\frac{1}{2}}}{n^2}\left[1 + O\left(\frac{k_n^{\frac{1}{2}}}{n}\right)\right] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{k_n^{\frac{1}{2}}}{n^2}\left(1 + o(1)\right).$$

To complete the proof of the theorem it is clearly sufficient from (5.1.6) and (5.1.7) and by the Schwarz inequality to show that

$$E\, H_n^2(U_{k_n}^{(n)}) = o\left(\frac{k_n^{1/2}}{n^2}\right).$$

Choose δ, $0 < \delta < \frac{1}{8}$. Let

$$\varepsilon_n = \frac{k_n^{\delta}}{n^{1/2}}$$

and $I_n = \left[\max\{x_0, \frac{k_n}{n+1} - \varepsilon_n\}, \frac{k_n}{n+1} + \varepsilon_n\right]$. Denoting the probability density of $U_{k_n}^{(n)}$ by g_n, we have

$$\begin{aligned} E\, H_n^2(U_{k_n}^{(n)}) &= \int_0^1 H_n^2(u) g_n(u)\, du \\ &= \left(\int_{u\in I_n} + \int_{u\notin I_n}\right) H_n^2(u) g_n(u)\, du. \end{aligned}$$

Let

$$H_{n,\max} = \sup_{u\in I_n} |H_n(u)|$$

and

$$g_{n,\max} = \sup_{u\notin I_n} g_n(u).$$

Then

$$\begin{aligned} E\, H_n^2(U_{k_n}^{(n)}) &\le H_{n,\max}^2 \int_{u\in I_n} g_n(u)\, du + g_{n,\max} \int_{u\notin I_n} H_n^2(u)\, du \\ &\le H_{n,\max}^2 + g_{n,\max} \int_0^1 H_n^2(u)\, du. \end{aligned}$$

Also, by the inequality $\frac{1}{3}(a+b+c)^2 \le a^2 + b^2 + c^2$ for real a, b, and c,

$$\begin{aligned} \frac{1}{3}\int_0^1 H_n^2(u)\, du &\le \int_0^1 Q^2(u)\, du + \int_0^1 Q^2\left(\frac{k_n}{n+1}\right) du + \int_0^1 \left[Q'\left(\frac{k_n}{n+1}\right)\right]^2 \left(u - \frac{k_n}{n+1}\right)^2 du \\ &\le E\xi_1^2 + Q^2\left(\frac{k_n}{n+1}\right) + \left[Q'\left(\frac{k_n}{n+1}\right)\right]^2. \end{aligned}$$

Since $E\xi_1^2 < \infty$, by (5.1.3) and (5.1.4), and since $F'(x_n')$ tends to a

non-zero limit, there exists a finite constant C_1 such that for $n \geq N_1$,

$$\int_0^1 H_n^2(u)\,du \leq C_1 \ .$$

Hence for $n \geq N_1$,

$$(5.1.8) \qquad E\, H_n^2\left(U_{k_n}^{(n)}\right) \leq H_{n,\max}^2 + C_1\, g_{n,\max} \ .$$

Since $\varepsilon_n \to 0$ we may choose $N_2 \geq N_1$ such that $I_n \in B$ for $n \geq N_2$. Then letting $C_2 = \max_{u \in B} |Q''(u)| < \infty$, we have by Taylor's expansion,

$$\left| Q(u) - Q\left(\frac{k_n}{n+1}\right) + \left(u - \frac{k_n}{n+1}\right) Q'\left(\frac{k_n}{n+1}\right) \right| \leq \frac{C_2}{2}\left(u - \frac{k_n}{n+1}\right)^2$$

for $u \in I_n$, and therefore

$$(5.1.9) \qquad H_{n,\max} \leq \frac{C_2}{2}\, \varepsilon_n^2$$

if $n \geq N_2$.

Now, we observe that the density

$$g_n(u) = \frac{n!}{(k_n-1)!\,(n-k_n)!}\, u^{k_n - 1}\, (1-u)^{n-k_n}$$

has mode $m_n = \dfrac{k_n - 1}{n-1}$ and decreases monotonically on both sides. Since

$$(5.1.10) \qquad \left| \frac{k_n}{n+1} - \frac{k_n - 1}{n-1} \right| = \frac{1 - \dfrac{2k_n}{n+1}}{n-1} = o(\varepsilon_n),$$

it is clear that $m_n \in I_n$ for all sufficiently large n. Let $\kappa \geq 1$. We have

$$E\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right)^{2\kappa} = \int_0^1 \left(u - \frac{k_n}{n+1}\right)^{2\kappa} g_n(u)\,du$$

$$\geq \int_{m_n}^{\frac{k_n}{n+1} + \varepsilon_n} \left(u - \frac{k_n}{n+1}\right)^{2\kappa} g_n(u)\,du$$

$$\geq g_n\left(\frac{k_n}{n+1} + \varepsilon_n\right) \int_{m_n}^{\frac{k_n}{n+1}+\varepsilon_n} \left(u - \frac{k_n}{n+1}\right)^{2\kappa} du$$

$$= g_n\left(\frac{k_n}{n+1} + \varepsilon_n\right)(2\kappa + 1)^{-1}\left[\varepsilon_n^{2\kappa+1} - \left(m_n - \frac{k_n}{n+1}\right)^{2\kappa+1}\right].$$

Since $n\varepsilon_n \to \infty$ whereas $m_n - \frac{k_n}{n+1} = O(n^{-1})$, by the first equality in (5.1.10), there exists $C_3 > 0$, depending on κ, such that

$$E\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right)^{2\kappa} \geq C_3\, g_n\left(\frac{k_n}{n+1} + \varepsilon_n\right) \varepsilon_n^{2\kappa+1} ,$$

or

$$g_n\left(\frac{k_n}{n+1} + \varepsilon_n\right) \leq E\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right)^{2\kappa} C_3^{-1}\, \varepsilon_n^{-(2\kappa+1)} ,$$

for sufficiently large n. In a similar manner there exists $C_4 > 0$ such that

$$g_n\left(\frac{k_n}{n+1} - \varepsilon_n\right) \leq E\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right)^{2\kappa} C_4^{-1}\, \varepsilon_n^{-(2\kappa+1)}$$

if $\frac{k_n}{n+1} - \varepsilon_n > 0$ infinitely often, and $g_n\left(\frac{k_n}{n+1} - \varepsilon_n\right) = 0$ if $\frac{k_n}{n+1} - \varepsilon_n \leq 0$ for all large n. Thus letting $C_5 = \min\{C_3, C_4\}$ in the former case and $C_5 = C_3$ in the latter, we have for $n \geq N_3 \geq N_2$,

$$g_{n,\max} \leq E\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right)^{2\kappa} C_5^{-1}\, \varepsilon_n^{-(2\kappa+1)} .$$

Since (see Blom (1958, p. 42)) there is a constant $C_6 < \infty$ independent of n, k_n , and κ such that

$$E\left(U_{k_n}^{(n)} - \frac{k_n}{n+1}\right)^{2} \leq C_6\, n^{-\kappa} ,$$

it follows that

(5.1.11) $$g_{n,\max} \leq C_6\, C_5^{-1}\, n^{-\kappa}\, \varepsilon_n^{-(2\kappa+1)}$$

for $n \geq N_3$. Combining (5.1.8), (5.1.9), and (5.1.11) then leads to

$$E\, H_n^2\left(U_{k_n}^{(n)}\right) \le \tfrac{1}{4}\, C_2^2 \varepsilon_n^4 + C_1 C_6 C_5^{-1}\, n^{-\kappa}\, \varepsilon_n^{-(2\kappa+1)}$$

for $n \ge N_3$. Finally, we may suppose that $k_n \ge n^{\theta}$ for some $\theta > 0$ and for all large n, so that

$$n^{-\kappa}\, \varepsilon_n^{-(2\kappa+1)} \le n^{-\kappa}\, (n^{\frac{1}{2}-\theta\delta})^{2\kappa+1} = n^{\frac{1}{2}-\theta\delta(2\kappa+1)} ,$$

and since κ is arbitrary, we may choose it sufficiently large to obtain

$$E\, H_n^2\left(U_{k_n}^{(n)}\right) = o\left(\frac{k_n^{\frac{1}{2}}}{n^2}\right),$$ which completes the proof.

5.2. Weak Convergence of Stochastic Processes.

In this section we introduce a stochastic process having a relationship to intermediate order statistics, analagous to that which the well-known "empirical process" has to sample λ-quantiles. There being substantial studies of the empirical process having been made and presented in the literature, with a large number of theoretical and applied results obtained, we conjecture that a corresponding extensive theory could be developed for the process we define. However, since the asymptotic distribution theory of intermediate order statistics has been our primary interest in this investigation, we study only the corresponding basic concept for stochastic processes, that of weak convergence, leaving open other potentially useful considerations.

We begin with a brief sketch of the background tools and terminology pertaining to weak convergence of stochastic processes. These are more fully discussed by Billingsley (1968).

Let $(D,\mathcal{D})$ be the metric space of all functions right-continuous on the unit interval $[0,1]$ and having left-hand limits, with the σ-field $\mathcal{D}$ generated by the Skorohod J_1-topology. If P_n , $n \ge 1$, and P are probability measures on $\mathcal{D}$, then P_n converges weakly to P,

denoted by $P_n \Rightarrow P$, if

$$\int_{x\in D} h(x)dP_n(x) \to \int_{x\in D} h(x)dP(x) \quad \text{as} \quad n \to \infty,$$

for every bounded J_1-continuous real-valued function h on D. If $X = \{X(t),\ 0\le t\le 1\}$ is a stochastic process such that each $X(t)$ is finite-valued, and having sample paths belonging to D, the distribution of X is the probability measure P_X on $\mathcal{D}$ given by

$$P_X(A) = PX^{-1}(A), \qquad A \in \mathcal{D}.$$

This is properly defined, since the mapping $X\colon (\Omega,\mathcal{F}) \to (D,\mathcal{D})$ which takes ω into the function $X(t,\omega)$, $0 \le t \le 1$, is measurable if and only if each $X(t)$ is a r.v. Correspondingly, a sequence $\{X_n(t),\ 0\le t\le 1\}$, $n \ge 1$, of stochastic processes is said to converge weakly, or in distribution, to the process $\{X(t),\ 0\le t\le 1\}$, denoted by $X_n \xrightarrow{D} X$, if $P_{X_n} \Rightarrow P_X$.

Now let ξ_n be an i.i.d. sequence of r.v.'s with marginal d.f. F and such that $0 \le \xi_1(\omega) \le 1$, and let $F_n(\cdot,\omega)$ be the empirical d.f. of $\xi_1,\dots,\xi_n$. The empirical process $Y_n = \{Y_n(t),\ 0\le t\le 1\}$ is defined by

$$Y_n(t,\omega) = n^{\frac{1}{2}}\big(F_n(t,\omega) - F(t)\big).$$

If $Y = \{Y(t),\ 0\le t\le 1\}$ is the normal process, which may be assumed to have sample paths belonging to D, defined by

$$EY(t) = 0$$

and

$$EY(s)Y(t) = F(s)\big(1 - F(t)\big), \qquad 0 \le s \le t \le 1,$$

then $Y_n \xrightarrow{D} Y$. (Billingsley, p. 141.) In particular, if the ξ_n are distributed uniformly on [0,1], so that $F(t) = t$, then Y is the Brownian Bridge process, which we may assume to have continuous sample paths.

Much of the interest in the empirical process is due to its usefulness, when appropriate bounded continuous functions h on D are considered, in deriving the asymptotic distributions of the Kolmogorov-Smirnov statistics, important in the theory of nonparametric inference, for arbitrary continuous marginal d.f.'s F (see, for example, Breiman (1968, p. 284) and Billingsley (1968, p. 104)). Also of interest is the familiar Glivenko-Cantelli theorem, a strong law of large numbers analogue for empirical processes, according to which the sample d.f.'s almost surely converge uniformly to the population (marginal) d.f. (Billingsley (1968, p. 103)). Other results include the law of the iterated logarithm for empirical processes (see, for example, Csörgö-Révésy (1974)) as well as extensions of these results to situations involving dependent sequences ξ_n (Billingsley (1968, p. 197), Deo (1973b)).

Having provided a few basic facts about the empirical process, we now proceed to define an analagous stochastic process and develop its weak convergence properties. Again let ξ_n be an i.i.d. sequence with marginal d.f. F. Suppose that F satisfies one of the following conditions:

a) F has left endpoint $x_0 = -\infty$ and is continuous in an interval $(-\infty,d)$.

b) F has a finite left endpoint x_0 such that $F(x_0) = 0$ and is

continuous in an interval $(x_0, x_0+\delta)$.

Let k_n be an intermediate rank sequence. Since $\frac{k_n}{n} \to 0$ there is an integer $N_0 \geq 1$ such that for $n \geq N_0$, $\frac{k_n}{n} < F(d-0)$ if F satisfies a), and $\frac{k_n}{n} < F(x_0 + \delta - 0)$ if F satisfies b). In either case there are numbers $x_n^{(t)}$ such that $F(x_n^{(t)}) = \frac{tk_n}{n}$, for each t, $0 < t \leq 1$, and for $n \geq N_0$. In general, we may define

$$x_n^{(t)} = \sup\left\{x\colon F(x) \leq \frac{tk_n}{n}\right\}.$$

for $0 \leq t \leq 1$ and for each $n \geq 1$.

For each $n \geq 1$ let $F_n(\cdot,\omega)$ be the empirical d.f. of $\xi_1,\ldots,\xi_n$. We define stochastic processes $X_n = \{X_n(t),\ 0\leq t\leq 1\}$ by

$$X_n(t,\omega) = \frac{n}{k_n^{\frac{1}{2}}}\left(F_n(x_n^{(t)},\ \omega) - \frac{tk_n}{n}\right). \tag{5.2.1}$$

Since for fixed n, $x_n^{(t)}$ as a function of t is right-continuous with left-hand limits on $[0,1]$, and actually is continuous if $n \geq N_0$, and since $F_n(\cdot,\omega)$ is in fact a d.f., it is clear that the sample paths of X_n are elements of D. Our object in this section is to show that $X_n \xrightarrow{D} W$, where $W = \{W(t),\ 0\leq t\leq 1\}$ is the standard Weiner process, which is characterized as the normal process with

$$EW(t) = 0$$

and

$$EW(s)W(t) = \min(s,t), \qquad 0 \leq s,\ t \leq 1.$$

As is well-known, W may be assumed to have continuous sample paths, and of course $W(0) = 0$ w.p.1.

Lemma 5.2.1. The finite dimensional distributions of X_n converge to those of W. That is, for given $0 \le t_1 < \dots < t_m \le 1$, the random vector $\left(X_n(t_1), \dots, X_n(t_m)\right)$ converges weakly to the random vector $\left(W(t_1), \dots, W(t_m)\right)$.

Proof: Since $X_n(0) = 0$ for all n, we may as well suppose that $0 < t_1 < \dots < t_m \le 1$. Let $c_1, \dots, c_m$ be real numbers. Then for each $n \ge N_0$ we have

$$c_1 X_n(t_1) + \dots + c_m X_n(t_m) = \frac{1}{k_n^{\frac{1}{2}}}\left\{c_1 \sum_{j=1}^{n} \left(I_{[\xi_j \le x_n(t_1)]} - \frac{t_1 k_n}{n}\right) + \dots + c_m \sum_{j=1}^{n} \left(I_{[\xi_j \le x_n(t_m)]} - \frac{t_m k_n}{n}\right)\right\}$$

$$= \sum_{j=1}^{n} X_{n,j} ,$$

say, where

$$X_{n,j} = \sum_{\mu=1}^{m} c_\mu \frac{I_{[\xi_j \le x_n(t_\mu)]} - \frac{t_\mu k_n}{n}}{k_n^{\frac{1}{2}}} .$$

Consider the array $\{X_{n,j}\}_{n \ge N_0, 1 \le j \le n}$. Since the sequence ξ_n is independent, so are the r.v.'s $X_{n,j}$, $j = 1, \dots, n$, for each n, and clearly $E X_{n,j} = 0$. Also we have

$$n \operatorname{Var} X_{n,j} = c_1^2 t_1\left(1 - \frac{t_1 k_n}{n}\right) + \dots + c_m^2 t_m\left(1 - \frac{t_m k_n}{n}\right) + \frac{2}{k_n} \sum_{1 \le \mu < \nu \le m} c_\mu c_\nu\, t_\mu k_n\left(1 - \frac{t_\nu k_n}{n}\right),$$

so that

$$n \operatorname{Var} X_{n,j} \to c_1^2 t_1 + \dots + c_m^2 t_m + 2 \sum_{1 \le \mu < \nu \le m} c_\mu c_\nu t_\mu$$

$$= \sigma^2 ,$$

say. Since $k_n \to \infty$ it is clear that $X_{n,j}$ satisfies the Lindeberg condition

$$\lim_n \sum_{i=1}^{n} E\left(X_{n,j}^2 \, I_{[|X_{n,j}|>\varepsilon]}\right) = 0 \quad \text{for every } \varepsilon > 0,$$

and hence $\sum_{j=1}^{n} X_{n,j}$ is asymptotically normal with mean zero and variance σ^2. That is, $\sum_{j=1}^{n} X_{n,j} \Rightarrow c_1 W(t_1) + \ldots + c_m W(t_m)$, where for $\mu = 1, \ldots, m$, $W(t_\mu)$ is normally distributed with mean zero and variance t_μ, and $EW(t_\mu)W(t_\nu) = t_\mu$ for $\mu < \nu$. Then by the Cramér-Wold device the lemma follows.

Following a procedure similar to that used for the empirical process, we may now obtain the main weak convergence result.

Theorem 5.2.2. Let ξ_n be i.i.d. with marginal d.f. F satisfying either condition a) or b) above, and let k_n be an intermediate rank sequence. Then for the sequence of stochastic processes X_n defined by (5.2.1), we have $X_n \xrightarrow{D} W$.

Proof: According to Theorem 15.6 of Billingsley (1968), in addition to the convergence of the finite dimensional distributions, in order that $X_n \xrightarrow{D} W$ it is sufficient that there be a nondecreasing continuous function f on $[0,1]$ such that

$$(5.2.2) \quad E|X_n(t) - X_n(t_1)|^2 |X_n(t_2) - X_n(t)|^2 \le \left(f(t_2) - f(t_1)\right)^2$$

for $0 \le t_1 \le t \le t_2 \le 1$ and for all (large) n.

Let $0 \le t_1 \le t \le t_2 \le 1$ and $n \ge N_0$ be given. Define

$$\alpha_i = I_{[x_n^{(t_1)} < \xi_i \le x_n^{(t)}]} - p_1$$

and

$$\beta_i = I_{\left[x_n^{(t)} < \xi_i \le x_n^{(t_2)}\right]} - p_2 ,$$

where $p_1 = P\left(x_n^{(t_1)} < \xi_1 \le x_n^{(t)}\right)$ and $p_2 = P\left(x_n^{(t)} < \xi_1 \le x_n^{(t_2)}\right)$. Then

$$X_n(t) - X_n(t_1) = \frac{1}{k_n^{\frac{1}{2}}} \sum_{i=1}^{n} \alpha_i$$

and

$$X_n(t_2) - X_n(t) = \frac{1}{k_n^{\frac{1}{2}}} \sum_{i=1}^{n} \beta_i .$$

Since the ξ_n are independent, and since $E\,\alpha_i = E\,\beta_i = 0$, $i = 1, \ldots,$ n, we have by straightforward calculations that

$$E\left(\sum_{i=1}^{n} \alpha_i\right)^2 \left(\sum_{i=1}^{n} \beta_i\right)^2 = n\,E\,\alpha_1^2\beta_1^2 + n(n-1)\,E\,\alpha_1^2\,E\,\beta_1^2$$
$$+ 2n(n-1)(E\,\alpha_1\beta_1)^2 .$$

Thus since

$$E\,\alpha_1^2\beta_1^2 = p_1(1-p_1)^2p_2^2 + p_2(1-p_2)^2p_1^2 + (1-p_1-p_2)p_1^2p_2$$
$$\le 3p_1p_2 ,$$

$$E\alpha_1^2 E\beta_1^2 = p_1(1-p_1)p_2(1-p_2) \le p_1p_2 ,$$

and

$$E\,\alpha_1\beta_1 = -p_1(1-p_1)p_2 - p_2(1-p_2)p_1 + (1-p_1-p_2)p_1p_2$$
$$= -p_1p_2 ,$$

we have

$$E\left(\sum_{i=1}^{n} \alpha_i\right)^2 \left(\sum_{\beta=1}^{n} \beta_i\right)^2 \le 6\,n^2\,p_1p_2 .$$

But $p_1 = (t - t_1)\frac{k_n}{n}$ and $p_2 = (t_2 - t)\frac{k_n}{n}$, so that

$$E\left(\sum_{i=1}^{n} \alpha_i\right)^2\left(\sum_{i=1}^{n} \beta_i\right)^2 \le 6\ k_n^2(t - t_1)(t_2 - t).$$

Hence

$$\begin{aligned} E|X_n(t) - X_n(t_1)|^2|X_n(t_2) - X_n(t)|^2 &\le 6(t - t_1)(t_2 - t) \\ &\le 6(t_2 - t_1)^2 . \end{aligned}$$

Therefore (5.2.2) holds with $f(t) = \sqrt{6}\ t$, completing the proof.

BIBLIOGRAPHY

Abramowitz, M., and Stegun, I.A. (1964). *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables.* New York, Dover Publications.

Bahadur, R.R. (1966). A note on quantiles in large samples. *The Annals of Mathematical Statistics, 37*, 577-580.

Balkema, A.A., and de Haan, L. (1972). On R. Von Mises' condition for the domain of attraction of $\exp(-e^{-x})$. *The Annals of Mathematical Statistics, 43*, 1352-1354.

van Beek, P. (1972). An application of Fourier methods to the problem of sharpening the Berry-Esseen inequality. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 23*, 187-196.

Berman, S.M. (1964). Limit theorems for the maximum term in stationary sequences. *The Annals of Mathematical Statistics, 35*, 502-516.

Billingsley, P. (1968). *Convergence of Probability Measures.* New York, John Wiley & Sons.

Blom, G. (1958). *Statistical Estimates and Transformed Beta-Varaibles.* New York, John Wiley & Sons.

Breiman, L. (1968). *Probability.* Reading, Massachusetts, Addison-Wesley Publishing Company.

Chen, P. (1966). The limiting distribution of the variation of a sequence of independent random variables. *Chinese Mathematics, 8*, 90-107.

Cheng, B. (1965). The limiting distributions of order statistics. *Chinese Mathematics, 6*, 84-104.

Chibisov, D.M. (1964). On limit distributions of order statistics. *Theory of Probability and Its Applications, 9*, 142-148.

Cramér, H., and Leadbetter, M.R. (1967). *Stationary and Related Stochastic Processes.* New York, John Wiley & Sons.

Csörgö, M., and Révész, P. (1974). Some notes on the empirical distribution function and the quantile process. *Limit Theorems of Probability, Colloquia Mathematica Societatis János Bolyai, 11.*

Deo, C.M. (1973a). A note on strong-mixing Gaussian sequences. *The Annals of Probability, 1*, 186-187.

________ (1973b). A note on empirical processes of strong-mixing sequences. *The Annals of Probability, 1*, 870-875.

Duttweiler, D.L. (1973). The mean-square error of Bahadur's order-statistic approximation. *The Annals of Statistics, 1*, 446-453.

Dvoretzky, A. (1972). Asymptotic normality for sums of dependent random variables. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, 2*, 513-535.

Feller, W. (1966). *An Introduction to Probability Theory and Its Applications, Volume II.* New York, John Wiley & Sons.

Fisher, R.A., and Tippett, L.H.C. (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Proceedings of the Cambridge Philosophical Society, 24*, 180-190.

Ghosh, J.K. (1971). A new proof of the Bahadur representation of quantiles and an application. *The Annals of Mathematical Statistics, 42*, 1957-1961.

Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics, 44*, 423-453.

de Haan, L. (1970). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes.* Mathematical Centre Tracts 7, Mathematisch Centrum, Amsterdam.

__________ (1971). A form of regular variation and its application to the domain of attraction of the double exponential. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 17*, 241-258.

Hoeffding, W., and Robbins, H. (1948). The central limit theorem for dependent random variables. *Duke Mathematical Journal, 15*, 773-780.

Kiefer, J. (1972). Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, 1*, 227-244.

Leadbetter, M.R. (1974a). *Lectures on Extreme Value Theory.* University of Lund.

____________ (1974b). On extreme values in stationary sequences. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 28*, 289-303.

____________ (1977). Extreme value theory under weak mixing conditions. Mathematical Association of America volume on probability, M. Rosenblatt, editor. To appear.

Leadbetter, M.R., Lindgren, G., de Maré, J., and Rootzén, H. (1977). Extremal and related properties of stationary processes. University of Umeå Technical Report, to appear.

Loève, M. (1956). Ranking limit problem. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 2*, 177-194.

________ (1963). *Probability Theory*, third edition. Princeton, D. Van Nostrand Company.

Loynes, R.M. (1965). Extreme values in uniformly mixing stationary stochastic processes. *The Annals of Mathematical Statistics, 36*, 993-999.

Miller, D.R. (1976). Order statistics, Poisson processes, and repairable systems. *Journal of Applied Probability, 13*, 519-529.

Mittal, Y., and Ylvisaker, D. (1976). Strong laws for the maxima of stationary Gaussian processes. *The Annals of Probability, 4*, 357-371.

Mori, T. (1976). Limit laws for maxima and second maxima from strong mixing processes. *The Annals of Probability, 4*, 122-126.

O'Brien, G.L. (1974). Limit theorems for the maximum term of a stationary process. *The Annals of Probability, 2*, 540-545.

Rosen, B. (1967). On the central limit theorem for sums of dependent random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 7*, 48-82.

Sen, P.K. (1968). Asymptotic normality of sample quantiles for m-dependent processes. *The Annals of Mathematical Statistics, 39*, 1724-1730.

________ (1972). On the Bahadur representation of sample quantiles for sequences of ϕ-mixing random variables. *Journal of Multivariate Analysis, 2*, 77-95.

Smirnov, N.V. (1952). The limit distributions for the terms of a variational series. *American Mathematical Society Translations*, Number 67.

__________ (1967). Some remarks on limit laws for order statistics. *Theory of Probability and Its Applications, 12*, 336-337.

Van Zwet, W.R. (1964). *Convex Transformations of Random Variables*. Mathematical Centre Tracts 7, Mathematisch Centrum, Amsterdam.

Welsch, R.E. (1971). A weak convergence theorem for order statistics from strong mixing processes. *The Annals of Mathematical Statistics, 42*, 1637-1646.

__________ (1972). Limit laws for extreme order statistics from strong mixing processes. *The Annals of Mathematical Statistics, 43*, 439-446.

Wilks, S.S. (1962). *Mathematical Statistics*. New York, John Wiley & Sons.

Wu, C.Y. (1966). The types of limit distributions for some terms of variational series. *Scientia Sinica, 15*, 749-762.